Black hole partition functions and duality

G.L. Cardoso[a], B. de Wit[b], J. Käppeli[c] and T. Mohaupt[d]

[a]*Arnold Sommerfeld Center for Theoretical Physics*
Department für Physik, Ludwig-Maximilians-Universität München, Munich, Germany
gcardoso@theorie.physik.uni-muenchen.de

[b]*Institute for Theoretical Physics* and *Spinoza Institute,*
Utrecht University, Utrecht, The Netherlands
B.deWit@phys.uu.nl

[c]*Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Potsdam, Germany*
kaeppeli@aei.mpg.de

[d]*Department of Mathematical Sciences, University of Liverpool, Liverpool L69 3BX, U.K.*
Thomas.Mohaupt@liverpool.ac.uk

ABSTRACT

The macroscopic entropy and the attractor equations for BPS black holes in four-dimensional $N = 2$ supergravity theories follow from a variational principle for a certain 'entropy function'. We present this function in the presence of R^2-interactions and non-holomorphic corrections. The variational principle identifies the entropy as a Legendre transform and this motivates the definition of various partition functions corresponding to different ensembles and a hierarchy of corresponding duality invariant inverse Laplace integral representations for the microscopic degeneracies.

Whenever the microscopic degeneracies are known the partition functions can be evaluated directly. This is the case for $N = 4$ heterotic CHL black holes, where we demonstrate that the partition functions are consistent with the results obtained on the macroscopic side for black holes that have a non-vanishing classical area. In this way we confirm the presence of a measure in the duality invariant inverse Laplace integrals. Most, but not all, of these results are obtained in the context of semiclassical approximations. For black holes whose area vanishes classically, there remain discrepancies at the semiclassical level and beyond, the nature of which is not fully understood at present.

1   Introduction

The degeneracy of BPS states for certain wrapped brane or string configurations, which can be identified with extremal black holes, defines a statistical or microscopic entropy. This statistical entropy can be successfully compared to the macroscopic entropy for the extremal black holes that arise as supersymmetric solutions of the effective field theory associated with the corresponding string compactification [1]. Initially, this comparison made use of the Bekenstein-Hawking area law for black hole entropy. More refined calculations [2, 3] of the asymptotic degeneracy of microstates revealed that there are corrections to the area law. Subsequently, it was demonstrated how higher-order derivative couplings based on chiral $N = 2$ superspace densities in the effective action account for a successful agreement with the microscopic results [4]. A necessary ingredient in this work is provided by Wald's definition of black hole entropy based on a Noether surface charge [5], which ensures the validity of the first law of black hole mechanics. This definition enabled the derivation of a general thermodynamic or macroscopic entropy formula for the $N = 2$ supergravity theories discussed above. Here a crucial role is played by the fixed-point behavior: at the black hole horizon supersymmetry enhancement forces some of the fields, and in particular the moduli, to fixed values determined by the electric and magnetic charges q and p carried by the black hole. This attractor phenomenon persists for the supergravity theories with higher-derivative interactions [6]. The macroscopic entropy is therefore a function solely of the black hole charges. Adopting this generalized notion of black hole entropy and the black hole attractor behaviour, agreement of the macroscopic entropy has been established with the known asymptotic microstate degeneracies to subleading order in the limit of large charges.

More recently these refinements have led to further insights and conjectures. In [7] it was observed that the thermodynamic entropy formula [4] including the full series of higher-derivative corrections can be rewritten as the Legendre transform of a real function $\mathcal{F}(p, \phi)$ with respect to the electrostatic potentials ϕ defined at the black hole horizon. The electric charges are retrieved by $q = \partial\mathcal{F}/\partial\phi$. Remarkably, the 'free energy' $\mathcal{F}(p, \phi)$ obtained in this way from the thermodynamic entropy is related to the topological string partition function $Z_{\text{top}}(p, \phi)$ by the simple relation (we use the normalizations of this paper)

$$\mathrm{e}^{\pi \, \mathcal{F}(p,\phi)} = |Z_{\text{top}}(p, \phi)|^2 \,. \tag{1.1}$$

According to the conjecture of [7], the function $\mathcal{F}$ on the left-hand side should be interpreted as the free energy associated with a black hole partition function defined in terms of the microscopic degeneracies $d(p, q)$, which for given charges p and q define the microcanonical partition function. In view of the above relation the black hole ensemble relevant for the comparison to topological strings is the one where the magnetic charges p and the electrostatic potentials ϕ are held fixed. With respect to the magnetic charges one is therefore dealing with a microcanonical ensemble, while the quantized electric charges are replaced by the continuous electrostatic potentials ϕ when passing to a canonical ensemble by a Laplace

transformation [7],

$$Z(p, \phi) = \sum_{\{q\}} d(p, q) \, e^{\pi \, q \cdot \phi} \, . \tag{1.2}$$

The conjecture is thus that the mixed microcanonical/canonical black hole partition function is given by

$$Z(p, \phi) \approx e^{\pi \, \mathcal{F}(p, \phi)} \, , \tag{1.3}$$

which, through (1.1), is related to the topological string.

As the effective action formed the starting point for the above conjecture, it is clear that there exists in any case an indirect relation with the topological string. The genus-g partition functions of the topological string [8] are known to be related to certain higher-derivative interactions in an $N = 2$ supersymmetric string effective action. The holomorphic anomaly associated with these partition functions is related to non-Wilsonian terms in the effective action associated with the propagation of massless states [9]. The crucial question is therefore to understand what the implications are of this conjecture beyond its connection to the effective action. Further work in that direction can be found in [10, 11, 12, 13], where the conjecture was tested for the case of non-compact Calabi-Yau spaces. Other work concerns the question of how the background dependence related to the holomorphic anomaly equations and how the wave function interpretation of the topological string partition functions are encoded in the black hole partition functions. Interesting progress in this direction can be found in [14, 15, 16].

Viewing $Z(p, \phi)$ as a holomorphic function in ϕ, the relation (1.3) can be used to express the microscopic black hole degeneracies as an inverse Laplace transform,

$$d(p, q) = \int d\phi \, Z(p, \phi) \, e^{-\pi \, q \cdot \phi} \approx \int d\phi \, e^{\pi [\mathcal{F}(p, \phi) - q \cdot \phi]} \, . \tag{1.4}$$

In the limit of large charges the result of the integral is expected to be equal to the exponent of the Legendre transform of $\pi \mathcal{F}$, which is, by definition, the macroscopic entropy that formed the starting point. The question is then whether (1.4) captures certain of the subleading corrections encoded in the microscopic degeneracies $d(p, q)$. Various results have been obtained to this extent, mostly for the case of 1/2-BPS black holes in $N = 4$ string theory [17, 18, 19, 20]. There are of course questions regarding the convergence of (1.2) and the required periodicity of $\exp[\pi \mathcal{F}(p, \phi)]$ under imaginary shifts in ϕ. The latter is, conversely, related to the necessity of having to specify integration contours for the complex ϕ-integrations when extracting black hole degeneracies from $\exp[\pi \mathcal{F}(p, \phi)]$ using (1.4).

While many questions seem to be primarily related to technical complications and must be discussed in a case-by-case fashion, the issue of covariance with respect to electric-magnetic duality transformations can be addressed in fairly broad generality. It forms the main subject of this paper. The status of electric-magnetic duality covariance of the original proposal (1.3) is at first somewhat obscured by the fact that one is working with a mixed canonical/microcanonical black hole ensemble and is therefore not treating the electric and magnetic

charges on equal footing. At first sight it is therefore not obvious what electric-magnetic duality covariance implies at the level of (1.3). Furthermore, the black hole degeneracies obtained through (1.4) should be consistent with duality symmetries such as S- or T-duality.

In this paper we start from the fully canonical black hole partition function depending on the electro- and magnetostatic potentials ϕ and χ, which are conjugate to the quantized electric and magnetic charges q and p,

$$Z(\phi, \chi) = \sum_{\{p,q\}} d(p, q) \, \mathrm{e}^{\pi[q \cdot \phi - p \cdot \chi]} \, . \tag{1.5}$$

Is there a function $\mathrm{e}^{2\pi \mathcal{H}(\chi, \phi)}$ that is (at least in semiclassical approximation) associated to $Z(\phi, \chi)$ in analogy to the original conjecture? If such a function exists, what is its relation to the function $\mathrm{e}^{\pi \mathcal{F}(p, \phi)}$ of (1.3)? The answers to these questions turn out to be intimately related to the existence of a variational principle for black hole entropy. The associated entropy function naturally accommodates both the higher-order derivative terms and certain non-holomorphic interactions. The strategy of this paper consists in uncovering this variational principle and thereby identifying $\mathrm{e}^{2\pi \mathcal{H}(\chi, \phi)}$. Then, using that $Z(p, \phi)$ is related to $Z(\phi, \chi)$ by an inverse Laplace transform with respect to χ, subleading corrections to the proposal (1.3) are derived at the semiclassical level. These corrections appear as measure factors when retrieving black hole degeneracies as in (1.4) and implement the requirement of covariance under electric-magnetic duality transformations.[1]

Our approach can be tested in cases where the microscopic degeneracies are known. This is the case for heterotic black holes in $N = 4$ supersymmetric string theory, where for the so-called CHL models the exact degeneracies of $1/4$- and $1/2$-BPS states are known. In the limit of large charges, the $1/4$-BPS states correspond to regular dyonic black holes carrying both electric and magnetic charges, whose area is much bigger than the string scale. Hence these black holes are called 'large'. The $1/2$-BPS black holes are either electrically or magnetically charged and their area is of order of the string scale. At the two-derivative level the effective action leads to a vanishing area. These black holes are called 'small'.

The exact dyon degeneracies are encoded in certain automorphic functions, from which both the asymptotic degeneracies and the dominant contributions to the partition function can be extracted. In this paper we demonstrate that in this way the microscopic data indeed yield the macroscopic results and, in particular, confirm the presence of the measure factors in integrals such as (1.4) and generalizations thereof. While these results are extremely satisfactory, we should stress that at present there is no evidence that the correspondence can be extended beyond the semiclassical level. Nevertheless the agreement in the dyonic case is impressive as it involves non-perturbative terms in the string coupling.

The microstates of the $1/2$-BPS black holes are the perturbative string states and their microcanonical partition function is therefore known. Here an analogous comparison with

---

[1] The consequences of this approach have been discussed by the authors at recent conferences. These include the 'Workshop on gravitational aspects of string theory' at the Fields Institute (Toronto, May 2005) and 'Strings 2005' (Toronto, July 2005; http://www.fields.utoronto.ca/audio/05-06/strings/wit/index.html). See also [21, 22, 23].

the macroscopic results turns out to be rather intricate and agreement is found at leading order only. In our opinion there is at this moment no satisfactory way to fully account for the relation between microscopic and macroscopic descriptions of the small black holes at the semiclassical level and beyond, in spite of the fact that partial successes have been reported. In this paper we note that the semiclassical description seems to depend sensitively on the higher-derivative corrections and even on the presence of the non-holomorphic corrections, so that reliable calculations are difficult. Beyond this observation we have not many clues as to what is actually responsible for the rather general lack of agreement, which is in such a sharp contrast to the situation encountered for the large black holes.

The outline of this paper is as follows. In section 2 the variational principle that underlies both the black hole attractor mechanism and black hole entropy is introduced. It is explained how to incorporate both the higher-derivative interactions as well as non-holomorphic interactions. In subsection 2.1 the variational principle is reformulated in terms of real coordinates. These real coordinates will correspond to the electro- and magnetostatic potentials measured at the black hole horizon. In section 3 the variational principle is worked out for the case when only a restricted set of variables is varied while the others are kept fixed at their attractor values. One thereby recovers, for example, the original observations of [7]. The various entropy functions obtained in this way are worked out for $N = 4$ models and the role of duality transformations and of non-holomorphic corrections is explained. In section 4 we identify the various free energies obtained from the variational principles with partition sums over corresponding ensembles involving the microscopic degeneracies. We generally prove that for cases where the semiclassical approximation is appropriate, the macroscopic and microscopic descriptions for large black holes are in agreement. In particular the effects of the proposed measure factors are elucidated for generic $N = 2$ models and subsequently worked out for the $N = 4$ examples under consideration. We describe the discrepancies that arise for small black holes. In section 5 the partition functions of dyonic 1/4-BPS and of 1/2-BPS black holes are considered for the CHL models. We review the agreement with the macroscopic results of the asymptotic degeneracies and present a direct calculation of the mixed partition function. The latter is in agreement with the measure factors derived earlier on the basis of the macroscopic results. However, the agreements only pertain to the large black holes. The troublesome features noted before for the small black holes persist here as well.

2 Macroscopic entropy as a Legendre transform

Lagrangians for $N = 2$ supergravity coupled to vector supermultiplets that depend at most quadratically on space-time derivatives of the fields, are encoded in a homogeneous holomorphic function $F(X)$ of second degree. Here the complex X^I are related to the vector multiplet scalar fields (henceforth called moduli), up to an overall identification by a complex space-time dependent factor. The Lagrangian does not depend on this function but only on its derivatives. The index $I = 0, 1, \ldots, n$ labels all the vector fields, including the graviphoton, so that the matter fields comprise n vector supermultiplets. The $(2n + 2)$-component vector

4

(X^I, F_I), whose components are sometimes called 'periods' in view of their connection with the periods of the holomorphic three-form of a Calabi-Yau three-fold, play a central role. Here F_I is defined by $F_I = \partial F/\partial X^I$. Under electric/magnetic duality transformations these periods rotate under elements of $\mathrm{USp}(n+1, n+1)$. It is possible to describe (X^I, F_I) as the holomorphic sections of a line bundle, but this is not needed below.

The function $F(X)$, and therefore the $F_I(X)$, can be modified by extra (holomorphic) terms associated with the so-called Weyl supermultiplet of supergravity. These modifications give rise to additional interactions involving higher space-time derivatives; the most conspicuous coupling is the one proportional to the square of the Weyl tensor. Furthermore the effective action will contain non-local interactions whose generic form has, so far, not been fully determined. These non-local interactions induce non-holomorphic terms in the $F_I(X)$ which are needed for realizing the invariance of the full effective action under symmetries that are not respected by the Wilsonian effective action. The latter action is based on holomorphic quantities and it describes the effect of integrating out massive degrees of freedom. Both these holomorphic and non-holomorphic modifications will play an important role in this paper.

Supersymmetric (BPS) black hole solutions exhibit the so-called attractor phenomenon [24, 25, 26], which implies that at the horizon the scalar moduli take values that are fixed in terms of the electric and magnetic charges of the black hole. Henceforth these charges will be denoted by q_I and p^I, respectively. Because the entropy is based on the horizon properties of the various fields, the attractor mechanism ensures that the macroscopic entropy can be expressed entirely in terms of the charges. The attractor equations originate from the fact that the BPS solutions exhibit supersymmetry enhancement at the horizon. Globally the BPS solution has residual $N = 1$ supersymmetry, but locally, at the horizon and at spatial infinity, the solution exhibits full $N = 2$ supersymmetry. Although the attractor mechanism was originally discovered in the supersymmetric context, it has been known for a while [27, 28] that it can also occur in a more general non-supersymmetric context. Recent studies discussing various aspects of this have appeared in [29, 30, 31, 32, 33, 34, 35, 36, 37]. For instance, in [29, 31] it was shown that the attractor mechanism also holds in the context of non-supersymmetric extremal black hole solutions in covariant higher-derivative gravity theories, provided one makes certain assumptions on the horizon geometry.

Since (X^I, F_I) and (p^I, q_I) transform identically under electric/magnetic duality, it is not surprising that the attractor equations define a linear relation between the period vector (X^I, F_I), its complex conjugate, and the charge vector (p^I, q_I). However, in view of the fact that the X^I are defined up to a complex rescaling it is clear that there should be a certain normalization factor whose behaviour under the rescalings is such that the resulting expression is invariant. This normalization can be absorbed into the definition of X^I and leads to the quantities Y^I and $F_I = \partial F(Y)/\partial Y^I$ which are no longer subject to these rescalings, although the function $F(Y)$ inherits, of course, the scaling properties of the original function $F(X)$ [38]. Performing the same rescaling to the square of the lowest component of the Weyl multiplet (this is an auxiliary tensor field that is usually called the graviphoton 'field

strength'), we obtain an extra complex scalar denoted by Υ. On the horizon the values of Y^I, F_I and Υ are fixed by the attractor equations,

$$Y^I - \bar{Y}^I = \mathrm{i}p^I \,, \qquad F_I(Y, \Upsilon) - \bar{F}_I(\bar{Y}, \bar{\Upsilon}) = \mathrm{i}q_I \,, \qquad \Upsilon = -64 \,. \tag{2.1}$$

Here we introduced a possible holomorphic dependence of the function F on the Weyl multiplet field Υ which will induce R^2 terms and other higher-derivative terms in the Wilsonian action. Supersymmetry requires the function $F(Y, \Upsilon)$ to be homogeneous of second degree,

$$F(\lambda Y, \lambda^2 \Upsilon) = \lambda^2 \, F(Y, \Upsilon) \,. \tag{2.2}$$

For the moment we ignore the issue of possible non-holomorphic corrections and first proceed in a holomorphic setting.

As stated in the introduction, the macroscopic black hole entropy follows from a variational principle. To see this, define the 'entropy function',

$$\Sigma(Y, \bar{Y}, p, q) = \mathcal{F}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) - q_I(Y^I + \bar{Y}^I) + p^I(F_I + \bar{F}_I) \,, \tag{2.3}$$

where p^I and q_I couple to the corresponding magneto- and electrostatic potentials at the horizon (cf. [6]) in a way that is consistent with electric/magnetic duality. The quantity $\mathcal{F}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$ will be denoted as the 'free energy' for reasons that will become clear. For the case at hand $\mathcal{F}$ is given by

$$\mathcal{F}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) = -\mathrm{i} \left(\bar{Y}^I F_I - Y^I \bar{F}_I \right) - 2\mathrm{i} \left(\Upsilon F_\Upsilon - \bar{\Upsilon} \bar{F}_\Upsilon \right) \,, \tag{2.4}$$

where $F_\Upsilon = \partial F / \partial \Upsilon$. Also this expression is compatible with electric/magnetic duality [39]. Varying the entropy function Σ with respect to the Y^I, while keeping the charges and Υ fixed, yields the result,

$$\delta\Sigma = \mathrm{i}(Y^J - \bar{Y}^I - \mathrm{i}p^I)\,\delta(F_I + \bar{F}_I) - \mathrm{i}(F_I - \bar{F}_I - \mathrm{i}q_I)\,\delta(Y^I + \bar{Y}^I) \,. \tag{2.5}$$

Here we made use of the homogeneity of the function $F(Y)$. Under the mild assumption that the matrix $N_{IJ} = 2\,\mathrm{Im}\,F_{IJ}$ is non-degenerate, it thus follows that stationary points of Σ satisfy the attractor equations.[2] Moreover, at the stationary point, we have $q_I Y^I - p^I F_I = -\mathrm{i}\left(\bar{Y}^I F_I - Y^I \bar{F}_I\right)$. The macroscopic entropy is equal to the entropy function taken at the attractor point. This implies that the macroscopic entropy is the Legendre transform of the free energy $\mathcal{F}$. An explicit calculation yields the entropy formula obtained in [4],

$$\mathcal{S}_{\mathrm{macro}}(p, q) = \pi \, \Sigma \Big|_{\mathrm{attractor}} = \pi \Big[|Z|^2 - 256 \,\mathrm{Im}\, F_\Upsilon \Big]_{\Upsilon = -64} \,, \tag{2.6}$$

where $|Z|^2 = p^I F_I - q_I Y^I$. Here the first term represents a quarter of the horizon area (in Planck units) so that the second term defines the deviation from the Bekenstein-Hawking area law. In view of the homogeneity properties and the fact that Υ takes a fixed value

---

[2]In the absence of Υ-dependent terms this variational principle was first proposed in [38]. Observe that it pertains specifically to black holes that exhibit supersymmetry enhancement at the horizon.

(namely the attractor value $\Upsilon = -64$), the second term will be subleading in the limit of large charges. Note, however, that also the area will contain subleading terms, as it will also depend on Υ. In the absence of Υ-dependent terms, the homogeneity of the function $F(Y)$ implies that the area scales quadratically with the charges. The Υ-dependent terms define subleading corrections to this result.

In the introduction we already mentioned that there exist black hole solutions whose horizon vanishes in the classical approximation [40, 41]. In that case the leading contribution to the macroscopic entropy originates entirely from R^2-interactions and scales only linearly with the charges. For example, this happens for black holes corresponding to certain perturbative heterotic $N = 4$ string states [42]. These black holes are called 'small' black holes in view of their vanishing classical area, while the generic ones are called 'large' black holes. We will adopt this terminology throughout this paper.

We now extend the above results to incorporate the non-holomorphic corrections. As it turns out [43], this extension is effected by changing the function $F(Y, \Upsilon)$ to $F(Y, \Upsilon) + 2\mathrm{i}\,\Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$, where Ω is real and homogeneous of second degree. When Ω equals the imaginary part of a holomorphic function of Y and Υ, so that Ω is harmonic, we can always absorb the holomorphic part into $F(Y, \Upsilon)$ and drop the anti-holomorphic part. Alternatively, this implies that all the Υ-dependent terms can always be absorbed into Ω and this observation will be exploited later on. The shift of the function F induces the following changes in the derivatives F_I, $\bar{F}_I$ and F_Υ,

$$F_I \to F_I + 2\mathrm{i}\,\Omega_I \,, \qquad F_\Upsilon \to F_\Upsilon + 2\mathrm{i}\,\Omega_\Upsilon \,, \tag{2.7}$$

where $\Omega_I = \partial\Omega/\partial Y^I$, $\Omega_{\bar{I}} = \partial\Omega/\partial\bar{Y}^I$ and $\Omega_\Upsilon = \partial\Omega/\partial\Upsilon$. Note that for holomorphic functions we do not use different subscripts (I and $\bar{I}$, or Υ and $\bar{\Upsilon}$, respectively) to distinguish holomorphic and anti-holomorphic derivatives. The homogeneity implies,

$$\begin{aligned} 2\,F - Y^I F_I &= 2\,\Upsilon F_\Upsilon \,, \\ 2\,\Omega - Y^I \Omega_I - \bar{Y}^I \Omega_{\bar{I}} &= 2\,\Upsilon\,\Omega_\Upsilon + 2\bar{\Upsilon}\,\Omega_{\bar{\Upsilon}} \,. \end{aligned} \tag{2.8}$$

Substituting (2.7) in the free energy $\mathcal{F}$, one obtains the following modified expression,

$$\begin{aligned} \mathcal{F}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) &= -\mathrm{i}\,(\bar{Y}^I F_I - Y^I \bar{F}_I) - 2\mathrm{i}\,(\Upsilon F_\Upsilon - \bar{\Upsilon}\bar{F}_\Upsilon) \\ &\quad + 4\,\Omega - 2(Y^I - \bar{Y}^I)(\Omega_I - \Omega_{\bar{I}}) \,. \end{aligned} \tag{2.9}$$

Here we made use of the second equation of (2.8). However, also the corresponding expression of the entropy function will be modified,

$$\Sigma(Y, \bar{Y}, p, q) = \mathcal{F}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) - q_I(Y^I + \bar{Y}^I) + p^I(F_I + \bar{F}_I + 2\mathrm{i}(\Omega_I - \Omega_{\bar{I}})) \,. \tag{2.10}$$

With this definition the variation of the entropy function induced by δY^I and $\delta\bar{Y}^I$ reads,

$$\begin{aligned} \delta\Sigma &= \mathrm{i}(Y^I - \bar{Y}^I - ip^I)\,\delta(F_I + \bar{F}_I + 2\mathrm{i}(\Omega_I - \Omega_{\bar{I}})) \\ &\quad - \mathrm{i}(F_I - \bar{F}_I + 2\mathrm{i}(\Omega_I + \Omega_{\bar{I}}) - iq_I)\,\delta(Y^I + \bar{Y}^I) \,, \end{aligned} \tag{2.11}$$

which is a straightforward generalization of (2.5). Its form confirms that Ω can be absorbed into the holomorphic F_I when Ω is harmonic. Stationary points of the modified entropy function thus satisfy the following attractor equations,

$$Y^I - \bar{Y}^I = \mathrm{i}p^I\,, \qquad \hat{F}_I - \bar{\hat{F}}_I = \mathrm{i}q_I\,, \tag{2.12}$$

where here and henceforth we use the notation $\hat{F}_I$ to indicate the modification by Ω,

$$\hat{F}_I = F_I + 2\mathrm{i}\,\Omega_I\,. \tag{2.13}$$

This leads to the definition of a modified period vector whose components consist of the Y^I and the $\hat{F}_I$, where the latter will, in general, no longer be holomorphic. This description for incorporating non-holomorphic corrections is in accord with the approach used in [44], where a function Ω was constructed for heterotic black holes by insisting that the (modified) periods transform consistently under S-duality. From the effective action point of view, the non-holomorphic contributions to Ω originate from the non-local invariants that must be included in the effective action. From the topological string point of view, these terms are related to the holomorphic anomaly which is due to a non-holomorphic dependence of the genus-g partition functions on the background [8, 45].

The issue of electric/magnetic duality is subtle in the presence of non-holomorphic corrections. We discuss it in subsection 3.1 when analyzing T- and S-duality for $N = 4$ heterotic black holes. S-duality requires the presence of non-holomorphic terms, which leads to an entropy function that is invariant under both T- and S-duality. To obtain the entropy one evaluates the entropy function at the attractor point. The result is precisely equal to (2.6) upon changing the function F into $F + 2\mathrm{i}\,\Omega$. Hence the entropy is the Legendre transform of the free energy (2.9).

In the following subsection 2.1 we consider the variational principle and the corresponding Legendre transform in terms of real variables corresponding to the electrostatic and magnetostatic potentials. This shows that the macroscopic entropy is in fact a Legendre transform of the so-called Hesse potential (or its appropriate extension).

2.1   A real basis and the Hesse potential

In this subsection we present a reformulation of the variational principle in terms of real variables. This allows us to find an interpretation of the full Legendre transform of the entropy in the context of special geometry. In special geometry one usually employs complex variables, but in the context of BPS solutions, it is the real and imaginary parts of the symplectic vector (Y^I, F_I) that play a role. Namely, the imaginary part is subject to the attractor equations, whereas the real part defines the electrostatic and magnetostatic potentials [6]. Therefore it is no surprise that the form of the variational formulae (2.3) and (2.5) suggests a formulation in terms of $2(n + 1)$ real variables equal to the potentials, rather than in terms of the real and imaginary parts of the $n + 1$ complex variables Y^I. The conversion between the two sets of coordinates is well-defined whenever $N_{IJ} = 2\,\mathrm{Im}\,F_{IJ}$ is non-degenerate. The discussion of

special geometry in terms of the real coordinates can be found in [46, 47]. It turns out that the prepotential of special geometry has a real counterpart [48], the Hesse potential, which is related to the imaginary part of the holomorphic prepotential by a Legendre transform [49]. The distinction between complex and real polarizations also played a role in the interpretation of the topological partition function as a wave function on moduli space [15]. The purpose of this subsection is to exhibit the variational principle in the context of these real variables and to show that the black hole entropy is just the Legendre transform of the (generalized) Hesse potential.

At this point we include the R^2-corrections encoded by Υ, but ignore the non-holomorphic correction, which will be dealt with later. The independent complex fields are (Y^I, Υ), and associated with them is the the holomorphic function $F(Y, \Upsilon)$, which is homogeneous of second degree (2.8). We start by decomposing Y^I and F_I into their real and imaginary parts,

$$Y^I = x^I + \mathrm{i}u^I \;, \qquad F_I = y_I + \mathrm{i}v_I \;, \tag{2.14}$$

where $F_I = F_I(Y, \Upsilon)$. The real parametrization is obtained by taking $(x^I, y_I, \Upsilon, \bar{\Upsilon})$ instead of $(Y^I, \bar{Y}^I, \Upsilon, \bar{\Upsilon})$ as the independent variables. Although Υ is a spectator, note that the inversion of $y_I = y_I(x, u, \Upsilon, \bar{\Upsilon})$ gives $\mathrm{Im}\, Y^I = u^I(x, y, \Upsilon, \bar{\Upsilon})$. To compare partial derivatives in the two parametrizations, we need (we refrain from explicitly indicating the Υ-dependence),

$$
\begin{aligned}
\left.\frac{\partial}{\partial x^I}\right|_u &= \left.\frac{\partial}{\partial x^I}\right|_y + \frac{\partial y_J(x,u)}{\partial x^I}\left.\frac{\partial}{\partial y_J}\right|_x \;, \\
\left.\frac{\partial}{\partial u^I}\right|_x &= \frac{\partial y_J(x,u)}{\partial u^I}\left.\frac{\partial}{\partial y_J}\right|_x \;, \\
\left.\frac{\partial}{\partial \Upsilon}\right|_{x,u} &= \left.\frac{\partial}{\partial \Upsilon}\right|_{x,y} + \frac{\partial y_I(x,u)}{\partial \Upsilon}\left.\frac{\partial}{\partial y_I}\right|_x \;.
\end{aligned}
\tag{2.15}
$$

The homogeneity will be preserved under the reparametrization in view of the fact that $y(x, u)$ is a homogeneous function of first degree. This is reflected in the equality,

$$
\begin{aligned}
&x^I \left.\frac{\partial}{\partial x^I}\right|_u + u^I \left.\frac{\partial}{\partial u^I}\right|_x + 2\,\Upsilon \left.\frac{\partial}{\partial \Upsilon}\right|_{x,u} + 2\,\bar{\Upsilon} \left.\frac{\partial}{\partial \bar{\Upsilon}}\right|_{x,u} \\
&= x^I \left.\frac{\partial}{\partial x^I}\right|_y + y_I \left.\frac{\partial}{\partial y_I}\right|_x + 2\,\Upsilon \left.\frac{\partial}{\partial \Upsilon}\right|_{x,y} + 2\,\bar{\Upsilon} \left.\frac{\partial}{\partial \bar{\Upsilon}}\right|_{x,y} \;.
\end{aligned}
\tag{2.16}
$$

It is straightforward to write down the inverse of (2.15),

$$
\begin{aligned}
\left.\frac{\partial}{\partial x^I}\right|_y &= \left.\frac{\partial}{\partial x^I}\right|_u + \frac{\partial u^J(x,y)}{\partial x^I}\left.\frac{\partial}{\partial u^J}\right|_x \;, \\
\left.\frac{\partial}{\partial y_I}\right|_x &= \frac{\partial u^J(x,y)}{\partial y_I}\left.\frac{\partial}{\partial u^J}\right|_x \;, \\
\left.\frac{\partial}{\partial \Upsilon}\right|_{x,y} &= \left.\frac{\partial}{\partial \Upsilon}\right|_{x,u} + \frac{\partial u^I(x,y)}{\partial \Upsilon}\left.\frac{\partial}{\partial u^I}\right|_x \;.
\end{aligned}
\tag{2.17}
$$

Combining (2.15) with (2.17) enables one to find explicit expressions for the derivatives of $y(x, u)$ and $u(x, y)$. One can easily verify that the reparametrization is not well defined when $\det(N_{IJ}) = 0$.

Since the Hesse potential occurring in special geometry is twice the Legendre transform of the imaginary part of the prepotential with respect to $u^I = \mathrm{Im}\, Y^I$, we define the generalized Hesse potential by

$$\mathcal{H}(x, y, \Upsilon, \bar{\Upsilon}) = 2\, \mathrm{Im}\, F(x + iu, \Upsilon, \bar{\Upsilon}) - 2\, y_I\, u^I \,, \tag{2.18}$$

which is a homogeneous function of second degree. With the help of (2.8) we find

$$\mathcal{H}(x, y, \Upsilon, \bar{\Upsilon}) = -\tfrac{1}{2}i(\bar{Y}^I F_I - \bar{F}_I Y^I) - i(\Upsilon F_\Upsilon - \bar{\Upsilon}\bar{F}_{\bar{\Upsilon}}) \,, \tag{2.19}$$

which is just proportional to the free energy defined in (2.4). However, while the term proportional to $\mathrm{Im}\,\Upsilon F_\Upsilon$ in (2.4) was introduced in order to obtain the correct attractor equations, this term is now a consequence of the natural definition (2.18), as we see explicitly in (2.19). It is gratifying to see that the corresponding variational principle thus has an interpretation in terms of special geometry. The entropy function (2.3) is now replaced by

$$\Sigma(x, y, p, q) = 2\,\mathcal{H}(x, y, \Upsilon, \bar{\Upsilon}) - 2\, q_I x^I + 2\, p^I y_I \,. \tag{2.20}$$

Indeed, extremization of Σ with respect to (x^I, y_I) yields

$$\frac{\partial \mathcal{H}}{\partial x^I} = q_I \,, \qquad \frac{\partial \mathcal{H}}{\partial y_I} = -p^I \,. \tag{2.21}$$

Using the relations (2.17) it is straightforward to show that the extremization equations (2.21) are just the attractor equations (2.1), written in terms of the new variables (x^I, y_I). Substituting (2.21) into Σ one can verify that the Legendre transform of $\mathcal{H}$ is proportional to the entropy (2.6),

$$
\begin{aligned}
\mathcal{S}_{\mathrm{macro}}(p, q) &= 2\pi\Big[\mathcal{H} - x^I \frac{\partial \mathcal{H}}{\partial x^I} - y_I \frac{\partial \mathcal{H}}{\partial y_I}\Big]_{\mathrm{attractor}} \\
&= 2\pi\Big[-\mathcal{H} + 2\,\Upsilon \frac{\partial \mathcal{H}}{\partial \Upsilon} + 2\,\bar{\Upsilon}\frac{\partial \mathcal{H}}{\partial \bar{\Upsilon}}\Big]_{\mathrm{attractor}} \,,
\end{aligned}
\tag{2.22}
$$

where we used the homogeneity of $\mathcal{H}$. This expression coincides with (2.6) as $\partial\mathcal{H}/\partial\Upsilon|_{x,y} = -iF_\Upsilon$.

Let us finally also include the non-holomorphic corrections. Using that $\mathcal{H}$ and Ω are homogeneous functions of second degree we find from (2.7) and (2.19) that adding the non-holomorphic corrections amounts to the replacement

$$\mathcal{H} \to \hat{\mathcal{H}} = \mathcal{H} + 2\Omega - (Y^I - \bar{Y}^I)(\Omega_I - \Omega_{\bar{I}}) \,. \tag{2.23}$$

Since $\mathcal{H}$ is the Legendre transform of $2\,\mathrm{Im}\, F$, we see that $\hat{\mathcal{H}}$ is the Legendre transform of $2\,\mathrm{Im}\, F(x+iu, \Upsilon, \bar{\Upsilon}) + 2\,\Omega(x, u, \Upsilon, \bar{\Upsilon})$, which is proportional to the non-holomorphic modification (2.9) of the free energy. However, $\hat{\mathcal{H}}$ is by definition a function of the shifted y_I-variables $\hat{y}_I = y_I + i(\Omega_I - \Omega_{\bar{I}})$. When using $(x^I, \hat{y}_I, \Upsilon, \bar{\Upsilon})$ as the independent variables, the variational principle and the attractor equations take the same form as before.

This observation fits with what is known about the complex and real polarization for the topological string. The holomorphic anomaly, which implies the existence of non-holomorphic

modifications of the genus-g topological free energies, is related to the fact that a complex parametrization of the moduli space requires the explicit choice of a complex structure [45]. If one instead chooses to parametrize the moduli space by real period vectors (the real polarization), then no explicit choice of a complex structure is required, and one arrives at a 'background independent' formulation [15]. Note, however, that the non-holomorphic terms of the complex parametrization are encoded in certain non-harmonic terms in the real parametrization.

Observe that there exists a two-form $\omega = \mathrm{d}x^I \wedge \mathrm{d}y_I$, which in special geometry is the symplectic form associated with the flat Darboux coordinates (x^I, y_I). This form is invariant under electric/magnetic duality. Possible R^2-corrections leave this two-form unaltered, whereas, in the presence of non-holomorphic corrections one expects that the appropriate extension will be given by $\omega = \mathrm{d}x^I \wedge \mathrm{d}\hat{y}_I$. The implication of this extension is not fully known, but this observation will play a role later on.

3   Partial Legendre transforms and duality

It is, of course, possible to define the macroscopic entropy as a Legendre transform with respect to only a subset of the fields, by substituting a subset of the attractor equations. This subset must be chosen such that the variational principle remains valid. These partial Legendre transforms constitute a hierarchy of Legendre transforms for the black hole entropy. We discuss two relevant examples, namely the one proposed in [7], where all the magnetic attractor equations are imposed, and the dilatonic one for heterotic black holes, where only two real potentials are left which together define the complex dilaton field [44]. At this stage, there is clearly no reason to prefer one version over the other. This will change in section 4 where we discuss corresponding partition functions and inverse Laplace transforms for the microscopic degeneracies.

One possible disadvantage of considering partial Legendre transforms is that certain invariances are no longer manifest. As it turns out, the dilatonic formulation does not suffer from this. The invariances of the dilatonic formulation are relegated to an additional subsection 3.1, where we also collect some useful formulae that we need in later sections.

Let us start and first impose the magnetic attractor equations so that only the real parts of the Y^I will be relevant. Hence one makes the substitution,

$$Y^I = \tfrac{1}{2}(\phi^I + \mathrm{i}p^I) \,. \tag{3.1}$$

The entropy function (2.3) then takes the form (for the moment we suppress non-holomorphic corrections),

$$\Sigma(\phi, p, q) = \mathcal{F}_{\mathrm{E}}(p, \phi, \Upsilon, \bar{\Upsilon}) - q_I \, \phi^I \,, \tag{3.2}$$

where the corresponding free energy $\mathcal{F}_{\mathrm{E}}(p, \phi)$ equals

$$\mathcal{F}_{\mathrm{E}}(p, \phi, \Upsilon, \bar{\Upsilon}) = 4 \, \mathrm{Im}\Big[F(Y, \Upsilon) \Big]_{Y^I = (\phi^I + \mathrm{i}p^I)/2} \,. \tag{3.3}$$

11

To show this one makes use of the homogeneity of the function $F(Y, \Upsilon)$.

When extremizing (3.2) with respect to ϕ^I we obtain the attractor equations $q_I = \partial \mathcal{F}_E / \partial \phi^I$. This shows that the macroscopic entropy is a Legendre transform of $\mathcal{F}_E(p, \phi)$ subject to $\Upsilon = -64$, as was first noted in [7]. The existence of this transformation motivated the conjecture that there is a relation with topological strings, in view of the fact that $\exp[\mathcal{F}_E]$ equals the modulus square of the topological string partition function (c.f. (1.1)).

Let us now introduce the non-holomorphic corrections to the above result, by starting from the entropy function (2.10) and comparing to (3.2). This leads to a modification of the expression (3.3) for $\mathcal{F}_E(p, \phi)$ [43],[3]

$$\mathcal{F}_E(p, \phi) = 4 \left[\operatorname{Im} F(Y, \Upsilon) + \Omega(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) \right]_{Y^I = (\phi^I + i p^I)/2}. \tag{3.4}$$

The form of the attractor equations, $q_I = \partial \mathcal{F}_E / \partial \phi^I$, remains unchanged and is equivalent to the second equation of (2.12). Note, however, that the electric and magnetic charges have been treated very differently in this case, so that duality invariances that involve both types of charges are hard to discuss.

Along the same line one can now proceed and eliminate some of the ϕ^I as well. A specific example of this, which is relevant in later sections, is the dilatonic formulation for heterotic black holes, where we eliminate all the ϕ^I with the exception of two of them which parametrize the complex dilaton field. This leads to an entropy function that depends only on the charges and on the dilaton field [44, 43]. We demonstrate some salient features below and in the next subsection 3.1. Here it is convenient to include all the Υ-dependent terms into Ω, which also contains the non-holomorphic corrections. The heterotic classical function $F(Y)$ is given by

$$F(Y) = -\frac{Y^1 Y^a \eta_{ab} Y^b}{Y^0}, \qquad a = 2, \dots, n, \tag{3.5}$$

with real constants η_{ab}. The function Ω depends only linearly on Υ and $\bar{\Upsilon}$, as well as on the complex dilaton field $S = -i Y^1 / Y^0$ and its complex conjugate $\bar{S}$. Imposing all the magnetic attractor equations yields,

$$\mathcal{F}_E(p, \phi) = \tfrac{1}{2}(S + \bar{S}) \left(p^a \eta_{ab} p^b - \phi^a \eta_{ab} \phi^b \right) - i(S - \bar{S}) \, p^a \eta_{ab} \phi^b + 4 \, \Omega(S, \bar{S}, \Upsilon, \bar{\Upsilon}), \tag{3.6}$$

where the dilaton field is expressed in the remaining fields ϕ^0 and ϕ^1 and the charges p^0, p^1 according to

$$S = \frac{-i \phi^1 + p^1}{\phi^0 + i p^0}. \tag{3.7}$$

Subsequently we impose the electric attractor equations for the q_a, which leads to a full determination of the Y^I in terms of the dilaton field,

$$Y^0 = \frac{\bar{P}(\bar{S})}{S + \bar{S}}, \qquad Y^1 = \frac{i S \, \bar{P}(\bar{S})}{S + \bar{S}}, \qquad Y^a = -\frac{\eta^{ab} \bar{Q}_b(\bar{S})}{2(S + \bar{S})}, \tag{3.8}$$

---

[3]Observe that this result cannot just be obtained by replacing the holomorphic function $F(Y, \Upsilon)$ by $F(Y, \Upsilon) + 2i \, \Omega$.

where we used $\eta^{ac}\eta_{cb} = \delta^a{}_b$, and we introduced the quantities,

$$
\begin{aligned}
Q(S) &= q_0 + \mathrm{i}Sq_1\,, \\
P(S) &= p^1 - \mathrm{i}Sp^0\,, \\
Q_a(S) &= q_a + 2\mathrm{i}S\,\eta_{ab}p^b\,.
\end{aligned}
\tag{3.9}
$$

Now the entropy function equals

$$
\Sigma(S,\bar{S},p^I,q_I) = \mathcal{F}_{\mathrm{D}}(S,\bar{S},p^I,q_a) - q_0\phi^0 - q_1\phi^1\,,
\tag{3.10}
$$

where

$$
\begin{aligned}
\mathcal{F}_{\mathrm{D}}(S,\bar{S},p^I,q_a) &= \mathcal{F}_{\mathrm{E}}(p,\phi) - q_a\phi^a \\
&= \frac{\frac{1}{2}q_a\eta^{ab}q_b + \mathrm{i}\,p^a q_a(S-\bar{S}) + 2\,|S|^2\,p^a\eta_{ab}p^b}{S+\bar{S}} + 4\,\Omega(S,\bar{S},\Upsilon,\bar{\Upsilon})\,, \tag{3.11}
\end{aligned}
$$

and

$$
q_0\phi^0 + q_1\phi^1 = \frac{2\,q_0p^1 - \mathrm{i}(q_0p^0 - q_1p^1)(S-\bar{S}) - 2\,q_1p^0\,|S|^2}{S+\bar{S}}\,.
\tag{3.12}
$$

Combining (3.11) with (3.12) yields,

$$
\Sigma(S,\bar{S},p,q) = -\frac{q^2 - \mathrm{i}p\cdot q\,(S-\bar{S}) + p^2\,|S|^2}{S+\bar{S}} + 4\,\Omega(S,\bar{S},\Upsilon,\bar{\Upsilon})\,,
\tag{3.13}
$$

where q^2, p^2 and $p\cdot q$ are T-duality invariant bilinears of the various charges, defined by

$$
\begin{aligned}
q^2 &= 2q_0p^1 - \tfrac{1}{2}q_a\eta^{ab}q_b\,, \\
p^2 &= -2p^0q_1 - 2p^a\eta_{ab}p^b\,, \\
q\cdot p &= q_0p^0 - q_1p^1 + q_ap^a\,.
\end{aligned}
\tag{3.14}
$$

These are the expressions that were derived in [44, 43]. The remaining attractor equations coincide with $\partial_S\Sigma(S,\bar{S},p,q) = 0$,

$$
\frac{q^2 + 2\mathrm{i}\,p\cdot q\,\bar{S} - p^2\,\bar{S}^2}{S+\bar{S}} + 4\,(S+\bar{S})\partial_S\Omega = 0\,.
\tag{3.15}
$$

Provided that Ω is invariant under S-duality, all the above equations are consistent with S- and T-duality as can be verified by using the transformation rules presented in the subsection below. As before, the value of $\Sigma(S,\bar{S},p,q)$ at the attractor point (including $\Upsilon = -64$) will yield the macroscopic entropy as a function of the charges. Also the entropy will then be invariant under T- and S-duality.

Finally we consider the quantity $\hat{K}$,

$$
\begin{aligned}
\hat{K} &= \mathrm{i}(\bar{Y}^I\hat{F}_I - Y^I\hat{\bar{F}}_I) \\
&= |Y^0|^2(S+\bar{S})\left[(T+\bar{T})^a\eta_{ab}(T+\bar{T})^b + \frac{2\,\partial_S\Omega}{(Y^0)^2} + \frac{2\,\partial_{\bar{S}}\Omega}{(\bar{Y}^0)^2}\right]\,,
\end{aligned}
\tag{3.16}
$$

where the S-duality invariant moduli T^a are defined by $T^a = -\mathrm{i}\,Y^a/Y^0$. Note that, by construction, $\hat{K}$ is invariant under T- and S-duality, as can be verified by using the transformations given in the subsection below.[4] At the attractor point $\hat{K}$ is proportional to the area, as follows from,

$$\hat{K}\Big|_{\text{attractor}} = |Z|^2 = -\frac{q^2 - \mathrm{i}p\cdot q\,(S - \bar{S}) + p^2\,|S|^2}{S + \bar{S}}\,, \tag{3.17}$$

subject to the attractor equation (3.15).

Heterotic BPS black holes can either be large or small. Small black holes have vanishing area at the two-derivative level, and they correspond to electrically charged 1/2-BPS states. When taking R^2-interactions into account, a horizon forms and also the entropy becomes non-vanishing. This phenomenon has been studied in more detail in [50, 51, 52]. Large black holes, on the other hand, have non-vanishing area at the two-derivative level. In models with $N = 4$ supersymmetry, they correspond to dyonic 1/4-BPS states.

3.1 Duality invariance and non-holomorphic corrections

In this subsection we demonstrate how the non-holomorphic terms enter in order to realize the invariance under certain duality symmetries. Here we follow the same strategy as in [44], but will consider a more extended class of models with $N = 4$ supersymmetry. In this work the language of $N = 2$ supergravity was used to establish the invariance under target-space duality (T-duality) and S-duality of black holes with $N = 4$ supersymmetry that arise in the toroidal compactification of heterotic string theory. This compactification leads to an effective $N = 4$ supergravity coupled to 22 abelian vector supermultiplets. Together with the 6 abelian graviphotons this leads to a total of 28 vector fields. As 4 of the graviphotons are absent in the truncation to $N = 2$ supergravity, the variables Y^I will be labeled by $I = 0, 1, \ldots, 23$. The central idea is to determine the entropy function in the context of $N = 2$ supergravity and to extend the charges at the end to 28 electric and 28 magnetic charges, by making use of T- and S-duality. However, there exist other $N = 4$ heterotic models based on modding out the theory by the action of some discrete abelian group, which can be discussed on a par. They correspond to a class of so-called CHL models [53], which have fewer than 28 abelian gauge fields. The $N = 2$ description is then based on a smaller number of fields Y^I, which we will specify in due course. At symmetry enhancement points in the respective moduli spaces the abelian gauge group is enlarged to a non-abelian one. All these models are dual to certain type-IIA string compactifications.

In the following we will discuss T- and S-duality for this class of models and describe their entropy functions. The $N = 2$ description is based on the holomorphic function (3.5), modified with Υ-dependent terms and possibly non-holomorphic corrections encoded in the

---

[4]Note that the invariance under T-duality is somewhat more subtle, as one can deduce immediately from the transformation of T^a under T-duality,

$$\delta T^a = \mathrm{i}\,b^a + c\,T^a + \mathrm{i}\,a_b\Big[-\tfrac{1}{2}\eta^{ab}\left(T^c\eta_{cd}T^d + 2\,(Y^0)^{-2}\partial_S\Omega\right) + T^a\,T^b\Big].$$

function Ω. In this case Ω depends only on Υ and on the dilaton field S, and their complex conjugates,

$$F = -\frac{Y^1 Y^a \eta_{ab} Y^b}{Y^0} + 2\mathrm{i}\,\Omega(S, \bar{S}, \Upsilon, \bar{\Upsilon})\,, \qquad a = 2, \ldots, n\,, \tag{3.18}$$

where the dilaton-axion field is described by $S = -\mathrm{i}\,Y^1/Y^0$, and η_{ab} is an $\mathrm{SO}(1, n-2)$ invariant metric of indefinite signature. The number n denotes the number of moduli fields, and is left unspecified for the time being. It is related to the rank of the gauge group that arises in the $N = 4$ compactification. The $\hat{F}_I$ associated with (3.18) are given by

$$
\begin{aligned}
\hat{F}_0 &= \frac{Y^1}{(Y^0)^2}\Big[Y^a \eta_{ab} Y^b - 2\,\partial_S \Omega\Big]\,, \\
\hat{F}_1 &= -\frac{1}{Y^0}\Big[Y^a \eta_{ab} Y^b - 2\,\partial_S \Omega\Big]\,, \\
\hat{F}_a &= -\frac{2\,Y^1}{Y^0}\,\eta_{ab} Y^b\,,
\end{aligned}
\tag{3.19}
$$

where we note the obvious constraint $Y^0 \hat{F}_0 + Y^1 \hat{F}_1 = 0$.

We now investigate under which condition the above function leads to T- and S-duality. In the case of a holomorphic function the period vector (Y^I, F_I) transforms in the usual way under symplectic transformations induced by electric/magnetic duality. When a subgroup of these symplectic transformations constitutes an invariance of the Wilsonian action, this implies that the transformations of the Y^I will induce precisely the correct transformations on the F_I. In the case of non-holomorphic terms one would like this to remain true so that the attractor equations will be consistent with the duality invariance. Following [44] we first turn to T-duality, whose infinitesimal transformations are given by

$$
\begin{aligned}
\delta Y^0 &= -c\,Y^0 - a_a\,Y^a\,, & \delta\hat{F}_0 &= c\,\hat{F}_0 + b^a\,\hat{F}_a\,, \\
\delta Y^1 &= -c\,Y^1 + \tfrac{1}{2}\eta^{ab} a_a\,\hat{F}_b\,, & \delta\hat{F}_1 &= c\,\hat{F}_1 + 2\eta_{ab} b^a\,Y^b\,, \\
\delta Y^a &= -b^a\,Y^0 + \tfrac{1}{2}\eta^{ab} a_b\,\hat{F}_1\,, & \delta\hat{F}_a &= a_a\,\hat{F}_0 + 2\eta_{ab} b^b\,Y^1\,,
\end{aligned}
\tag{3.20}
$$

where the a_a, b^a and c denote $2n-1$ infinitesimal transformation parameters; upon combining these transformations with the obvious $\mathrm{SO}(1, n-2)$ transformations that act linearly on the Y^a (and on the $\hat{F}_a$), one obtains the group $\mathrm{SO}(2, n-1)$. Note that the dilaton field S is invariant under T-duality, while $(Y^0, \hat{F}_1, Y^a)$ and $(\hat{F}_0, -Y^1, \hat{F}_a)$ transform both as vectors under $\mathrm{SO}(2, n-1)$. It can now be verified that the variations $\delta\hat{F}_I$ are precisely induced by the variations δY^I, irrespective of the precise form of $\Omega(S, \bar{S}, \Upsilon, \bar{\Upsilon})$.

Under *finite* S-duality transformations, the situation is more complicated. Here the Y^I transform as follows,

$$
\begin{aligned}
Y^0 &\rightarrow d\,Y^0 + c\,Y^1\,, \\
Y^1 &\rightarrow a\,Y^1 + b\,Y^0\,, \\
Y^a &\rightarrow d\,Y^a - \tfrac{1}{2}c\,\eta^{ab}\,\hat{F}_b\,,
\end{aligned}
\tag{3.21}
$$

where a, b, c, d are integers, or belong to a subset of integers that parametrize a subgroup of $\mathrm{SL}(2; \mathbb{Z})$, and satisfy $ad - bc = 1$. As a result of these transformations, S transforms according

to the well-known formulae,

$$S \to S' = \frac{aS - \mathrm{i}b}{\mathrm{i}cS + d}, \qquad \frac{\partial S'}{\partial S} = \frac{1}{(\mathrm{i}c\,S + d)^2}. \tag{3.22}$$

When applied to the $\hat{F}_I$ these transformations induce the changes,

$$\begin{aligned}
\hat{F}_0 &\to a\,\hat{F}_0 - b\,\hat{F}_1 + \Delta_0\,, \\
\hat{F}_1 &\to d\,\hat{F}_1 - c\,\hat{F}_0 + \Delta_1\,, \\
\hat{F}_a &\to a\,\hat{F}_a - 2b\,\eta_{ab}Y^b\,,
\end{aligned} \tag{3.23}$$

where Δ_0 and Δ_1 are proportional to the same expression,

$$\Delta_0 \propto \Delta_1 \propto \partial_{S'}\Omega(S', \bar{S}', \Upsilon, \bar{\Upsilon}) - (\mathrm{i}c\,S + d)^2\,\partial_S\Omega(S, \bar{S}, \Upsilon, \bar{\Upsilon})\,, \tag{3.24}$$

which vanishes when $\partial_S\Omega$ is a modular form of weight two [44]. However, it is well known that there exists no modular form of weight two. In order to have attractor equations that transform covariantly under S-duality we are therefore forced to include non-holomorphic expressions. In applying this argument one may have to restrict Υ to its attractor value, but subject to this restriction Ω must be invariant under S-duality. Once the S-duality group is specified, the form of Ω will usually follow uniquely.

Observe that the duality transformations of the charges follow directly from those of the periods. In particular, the charge vectors (p^0, q_1, p^a) and $(q_0, -p^1, q_a)$ transform irreducibly under the T-duality group. The three T-duality invariants (3.14) transform as a vector under the S-duality group. Furthermore the quantities (3.9) transform under S-duality as a modular function,

$$(Q(S), P(S), Q_a(S)) \longrightarrow \frac{1}{\mathrm{i}c\,S + d}\,(Q(S), P(S), Q_a(S))\,, \tag{3.25}$$

and as a vector under T-duality.

We will now discuss the expressions for Ω for the class of CHL models [53] discussed recently in [54, 19, 55]. First we introduce the unique cusp forms of weight $k + 2$ associated with the S-duality group $\Gamma_1(N) \subset \mathrm{SL}(2; \mathbb{Z})$, defined by

$$f^{(k)}(S) = \eta^{k+2}(S)\,\eta^{k+2}(NS)\,, \tag{3.26}$$

where N is a certain positive integer. Hence these cusp forms transform under the S-duality transformations that belong to the subgroup $\Gamma_1(N)$, according to

$$f^{(k)}(S') = (\mathrm{i}c\,S + d)^{k+2}\,f^{(k)}(S)\,. \tag{3.27}$$

The subgroup $\Gamma_1(N)$ requires the transformation parameters to be restricted according to $c = 0 \mod N$ and $a, d = 1 \mod N$, which is crucial for deriving the above result. The integers k and N are not independent in these models and subject to

$$(k + 2)(N + 1) = 24\,. \tag{3.28}$$

The values $k = 10$ and $N = 1$ correspond to the toroidal compactification. Following [55], we will restrict attention to the values $(N, k) = (1, 10), (2, 6), (3, 4), (5, 2)$ and $(7, 1)$. The rank of the gauge group (corresponding to the number of abelian gauge fields in the effective supergravity action) is then equal to $r = 28, 20, 16, 12$ or 10, respectively. The corresponding number of $N = 2$ matter vector supermultiplets is then given by $n = 2(k + 2) - 1$.

The function Ω can now be expressed in terms of the cusp forms,

$$\Omega_k(S, \bar{S}, \Upsilon, \bar{\Upsilon}) = \frac{1}{256\,\pi} \left[\Upsilon \log f^{(k)}(S) + \bar{\Upsilon} \log f^{(k)}(\bar{S}) + \tfrac{1}{2}(\Upsilon + \bar{\Upsilon}) \log(S + \bar{S})^{k+2} \right], \quad (3.29)$$

in close analogy to the case $k = 10$ [44, 43]. Note that these terms agree with the terms obtained for the corresponding effective actions (see, for instance, [56, 57]). Suppressing instanton corrections this result takes the form,

$$\Omega_k(S, \bar{S}, \Upsilon, \bar{\Upsilon}) \Big|_{\Upsilon = -64} = \tfrac{1}{2}(S + \bar{S}) - \frac{k + 2}{4\pi} \log(S + \bar{S}). \quad (3.30)$$

This implies that, in the limit of large charges, the entropy of small black holes (with vanishing charges $p^0, q_1, p^2, \ldots, p^n$) will be independent of k and its leading contribution will be equal to one-half of the area. The latter follows from the entropy function (3.13) which, in this case, reads,

$$\Sigma(S, \bar{S}, p, q) = -\frac{q^2}{S + \bar{S}} + 2(S + \bar{S}) - \frac{k + 2}{\pi} \log(S + \bar{S}). \quad (3.31)$$

Stationarity of Σ shows that $S + \bar{S} \approx \frac{1}{4\pi}(k + 2) + \sqrt{|q^2|/2}$, while the entropy $S_{\text{macro}} \approx 4\pi\sqrt{|q^2|/2} - \tfrac{1}{2}(k+2) \log |q^2|$. The logarithmic term is related to the non-holomorphic term in (3.29), and its coefficient is not in agreement with microstate counting. However, this term is subject to semiclassical corrections, as we will discuss in the following sections. In the same approximation the area equals $8\pi\sqrt{|q^2|/2}$.

4 Partition functions and inverse Laplace transforms

So far, we discussed black hole entropy from a macroscopic point of view. To make the connection with microstate degeneracies, we conjecture, in the spirit of [7], that the Legendre transforms of the entropy are indicative of a thermodynamic origin of the various entropy functions. It is then natural to assume that the corresponding free energies are related to black hole partition functions corresponding to suitable ensembles of black hole microstates. To examine the consequences of this idea, let us define the following partition function,

$$Z(\phi, \chi) = \sum_{\{p, q\}} d(p, q)\, e^{\pi[q_I \phi^I - p^I \chi_I]}, \quad (4.1)$$

where $d(p, q)$ denotes the microscopic degeneracies of the black hole microstates with black hole charges p^I and q_I. This is the partition sum over a canonical ensemble, which is invariant under the various duality symmetries, provided that the electro- and magnetostatic potentials (ϕ^I, χ_I) transform as a symplectic vector. Identifying a free energy with the logarithm of $Z(\phi, \chi)$ it is clear that it should, perhaps in an appropriate limit, be related to the macroscopic

free energy introduced earlier. On the other hand, viewing $Z(\phi, \chi)$ as an analytic function in ϕ^I and χ_I, the degeneracies $d(p, q)$ can be retrieved by an inverse Laplace transform,

$$d(p, q) \propto \int d\chi_I \, d\phi^I \, Z(\phi, \chi) \, e^{\pi[-q_I\phi^I + p^I\chi_I]} \,, \tag{4.2}$$

where the integration contours run, for instance, over the intervals $(\phi-\mathrm{i}, \phi+\mathrm{i})$ and $(\chi-\mathrm{i}, \chi+\mathrm{i})$ (we are assuming an integer-valued charge lattice). Obviously, this makes sense as $Z(\phi, \chi)$ is formally periodic under shifts of ϕ and χ by multiples of 2i.

All of the above arguments suggest to identify $Z(\phi, \chi)$ with the generalized Hesse potential, introduced in subsection 2.1,

$$\sum_{\{p,q\}} d(p, q) \, e^{\pi[q_I\phi^I - p^I\hat{\chi}_I]} \sim \sum_{\text{shifts}} e^{2\pi \, \mathcal{H}(\phi/2, \hat{\chi}/2, \Upsilon, \bar{\Upsilon})} \,, \tag{4.3}$$

where Υ is equal to its attractor value and where the definition of $\hat{\chi} = 2\,\hat{y}$ was explained in subsection 2.1. Because the generalized Hesse potential is a macroscopic quantity which does not in general exhibit the periodicity that is characteristic for the partition function, the right-hand side of (4.3) requires an explicit periodicity sum over discrete imaginary shifts of the ϕ and χ. In case that the Hesse potential exhibits a certain periodicity (say, with a different periodicity interval), then the sum over the imaginary shifts will have to be modded out appropriately such as to avoid overcounting. This is confirmed by the result of the calculation we will present in subsection 5.2.1. At any rate, we expect that when substituting $2\pi\mathcal{H}$ into the inverse Laplace transform, the periodicity sum can be incorporated into the integration contours.

Unfortunately, it is in general difficult to find an explicit representation for the Hesse potential, as the relation (2.14) between the complex variables Y^I and the real variables x^I and y_I is complicated. Therefore we rewrite the above formulae in terms of the complex variables Y^I, where explicit results are known. In that case the relation (4.3) takes the following form,

$$\sum_{\{p,q\}} d(p, q) \, e^{\pi[q_I(Y+\bar{Y})^I - p^I(\hat{F}+\bar{\hat{F}})_I]} \sim \sum_{\text{shifts}} e^{\pi \, \mathcal{F}(Y,\bar{Y},\Upsilon,\bar{\Upsilon})} \,, \tag{4.4}$$

where $\mathcal{F}$ equals the free energy (2.9). Here we note that according to (2.5) the natural variables on which $\mathcal{F}$ depends, are indeed the real parts of Y^I and $\hat{F}_I$. Needless to say, the relation (4.4) (and its preceding one) is rather subtle, but it is reassuring that both sides are manifestly consistent with duality.

Just as indicated in (4.2), it is possible to formally invert (4.4) by means of an inverse Laplace transform,

$$\begin{aligned} d(p, q) & \propto \int d(Y+\bar{Y})^I \, d(\hat{F}+\bar{\hat{F}})_I \, e^{\pi \, \Sigma(Y,\bar{Y},p,q)} \\ & \propto \int dY \, d\bar{Y} \, \Delta^-(Y,\bar{Y}) \, e^{\pi \, \Sigma(Y,\bar{Y},p,q)} \,, \end{aligned} \tag{4.5}$$

where $\Delta^-(Y, \bar{Y})$ is an integration measure whose form depends on $\hat{F}_I + \hat{\bar{F}}_I$. The expression for Δ^- follows directly from (2.13). We also define a similar determinant Δ^+ that we shall need shortly,

$$\Delta^\pm(Y, \bar{Y}) = \left| \det \left[\mathrm{Im}\, F_{KL} + 2 \,\mathrm{Re}(\Omega_{KL} \pm \Omega_{K\bar{L}}) \right] \right| . \tag{4.6}$$

Here we note the explicit dependence on Ω. As before, F_{IJ} and F_I refer to Y-derivatives of the holomorphic function $F(Y, \Upsilon)$ whereas Ω_{IJ} and $\Omega_{I\bar{J}}$ denote the holomorphic and mixed holomorphic-antiholomorphic second derivatives of Ω, respectively.

It is not a priori clear whether the integral (4.5) is well-defined. Although the integration contours can in principle be deduced from the contours used in (4.2), an explicit determination is again not possible in general. Of course, the contours can be deformed but this depends crucially on the integrand whose analytic structure is a priori not clear. Here, it is important to realize that the analytic continuation refers to the initial variables in the inverse Laplace transform (4.2), provided by the electro- and magnetostatic potentials. Therefore the analytic continuation does not automatically respect the relation between Y^I and $\bar{Y}^I$ based on complex conjugation. Just as before, the effect of a periodicity sum on the right-hand side of (4.4) can be incorporated into the integration contour, but the periodicity sum is also defined in terms of the original variables. Obviously these matters are rather subtle and can only be addressed in specific models. A separate requirement is that the integration contours should be consistent with duality. Here it is worth pointing out that explicit integral representations for microscopic black hole degeneracies are known, although their (auxiliary) integration parameters have no direct macroscopic significance, unlike in (4.5). These representations will shortly play an important role.

Leaving aside these subtle points we will first establish that the integral representation (4.5) makes sense in case that a saddle-point approximation is appropriate. In view of the previous results it is clear that the saddle point coincides with the attractor point, so that the integrand should be evaluated on the attractor point. Evaluating the second variation of Σ,

$$
\begin{aligned}
\delta^2 \Sigma \;=\;& \mathrm{i}(Y^I - \bar{Y}^I - \mathrm{i}p^I)\, \delta^2(F_I + \bar{F}_I + 2\mathrm{i}(\Omega_I - \Omega_{\bar{I}})) \\
& + 2\mathrm{i} \left(\delta Y^I \, \delta(\bar{F}_I - 2\mathrm{i}\Omega_{\bar{I}}) - \delta(F_I + 2\mathrm{i}\Omega_I)\, \delta \bar{Y}^I \right) ,
\end{aligned}
\tag{4.7}
$$

and imposing the attractor equations so that $\delta\Sigma = 0$, one expands the exponent around the saddle point and evaluates the semiclassical Gaussian integral. This integral leads to a second determinant which, when $Y^I - \bar{Y}^I - \mathrm{i}p^I = 0$, factorizes into the square roots of two subdeterminants, $\sqrt{\Delta^+}$ and $\sqrt{\Delta^-}$. Here the plus (minus) sign refers to the contribution of integrating over the real (imaginary) part of δY^I. Consequently, the result of a saddle-point approximation applied to (4.5) yields,

$$d(p, q) = \sqrt{\left| \frac{\Delta^-(Y, \bar{Y})}{\Delta^+(Y, \bar{Y})} \right|}\Bigg|_{\mathrm{attractor}} \, e^{\mathcal{S}_{\mathrm{macro}}(p,q)} . \tag{4.8}$$

In the absence of non-holomorphic corrections the ratio of the two determinants is equal to unity and one thus recovers precisely the macroscopic entropy. Furthermore one can easily

convince oneself that the saddle-point approximation leads to results that are compatible with duality.

Before discussing this result in more detail, let us also consider the case where one integrates only over the imaginary values of δY^I in saddle-point approximation. The saddle point then occurs in the subspace defined by the magnetic attractor equations, so that one obtains a modified version of the OSV integral [7],

$$d(p,q) \propto \int \, \mathrm{d}\phi \, \sqrt{\Delta^-(p,\phi)} \, \mathrm{e}^{\pi[\mathcal{F}_{\mathrm{E}}(p,\phi) - q_I \phi^I]} \;, \tag{4.9}$$

where $\mathcal{F}_{\mathrm{E}}(p,\phi)$ was defined in (3.4) and $\Delta^-(p,\phi)$ is defined in (4.6) with the Y^I given by (3.1). Hence this integral must contain a measure factor $\sqrt{\Delta^-}$ in order to remain consistent with electric/magnetic duality.[5] Without the measure factor this is the integral conjectured by [7]. Inverting this formula to a partition sum over a mixed ensemble, one finds,

$$Z(p,\phi) = \sum_{\{q\}} d(p,q) \, \mathrm{e}^{\pi \, q_I \phi^I} \sim \sum_{\mathrm{shifts}} \sqrt{\Delta^-(p,\phi)} \, \mathrm{e}^{\pi \, \mathcal{F}_{\mathrm{E}}(p,\phi)} \;. \tag{4.10}$$

However, we note that this expression and the preceding one is less general than (4.5) because it involves a saddle-point approximation. Moreover the function $\mathcal{F}_{\mathrm{E}}$ is not duality invariant and the invariance is only recaptured when completing the saddle-point approximation with respect to the fields ϕ^I. Therefore one expects that an evaluation of (4.9) beyond the saddle-point approximation will entail a violation of (some of) the duality symmetries again, because it amounts to an unequal treatment of the real and the imaginary parts of the Y^I. Hence the situation regarding (4.9) and (4.10) remains unsatisfactory.

In [7], the partition function $Z(p,\phi)$ was conjectured to be given by the modulus square of the partition function of the topological string. This equality holds provided $\mathcal{F}_{\mathrm{E}}$ does not contain contributions from non-holomorphic terms, and provided there is no nontrivial multiplicative factor. The above observation has inspired further interest in the relation between the holomorphic anomaly equation of topological string theory and possible contributions from non-holomorphic terms to the black hole entropy (see the work quoted in the introduction). However, as we already mentioned in the introduction, a known relationship exists via the non-Wilsonian part of the effective action. For instance, as shown in [9], the holomorphic anomaly of topological string partition functions is precisely related to the non-local part of the action induced by massless string states.

Obviously, one can test the underlying conjecture by calculating the inverse Laplace transforms (4.5) and (4.9), using the macroscopic data as input and comparing with the known asymptotic degeneracies. Another approach is to start instead from known microscopic degeneracies and determine the partition functions (4.4) or (4.10), which can then be compared to the macroscopic data. Unfortunately there are not many examples available where one knows both macroscopic and microscopic results. In the remainder of this paper we will therefore restrict ourselves to the case of heterotic black holes with $N = 4$ supersymmetry,

---

[5]There has been some discussion in the literature about a possible modification of this integral, such as for instance by a measure factor [20, 58, 59]. See also footnote 1.

which we already introduced from a macroscopic perspective in section 3. Although there are positive results, many intriguing questions remain. A particular pertinent question concerns the domain of validity of this approach, which, unfortunately, we will not be able to answer.

In section 5 we will approach the comparison from the microscopic side, while in this section we will start from the macroscopic side and examine a number of results based on the inverse Laplace transforms (4.5) and (4.9). A number of tests based on (4.9), without including the measure factor Δ^- and the non-holomorphic contributions, have already appeared in the literature [17, 18, 19, 20]. These concern the electric (small) black holes. However, let us first discuss the more generic case of large black holes and evaluate the determinants $\Delta^\pm$ for arbitrary charge configurations. Some of the relevant expressions were already presented in section 3 and we use them to evaluate the determinants (4.6). The result reads as follows,

$$\Delta^\pm = \frac{(S + \bar{S})^{n-3} \det[-\eta_{ab}]}{4\,|Y^0|^4} \left[\left(\hat{K} \pm 2\,(S + \bar{S})^2 \partial_S \partial_{\bar{S}} \Omega \right)^2 - 4\,\left| (S + \bar{S})^2\, D_S \partial_S \Omega \right|^2 \right], \quad (4.11)$$

where $\hat{K}$ has been defined in (3.16) and

$$D_S \partial_S \Omega = \partial_S \partial_S \Omega + \frac{2}{S + \bar{S}} \partial_S \Omega \,. \tag{4.12}$$

Provided that Ω is invariant under S-duality, also $(S + \bar{S})^2 \partial_S \partial_{\bar{S}} \Omega$ and $|(S + \bar{S})^2\, D_S \partial_S \Omega|$ are invariant. This is confirmed by the fact that the measure $(S + \bar{S})^{n-3}|Y^0|^{-4} \prod_I \mathrm{d}Y^I \, \mathrm{d}\bar{Y}^I$ factorizes into two parts, $[|Y^0|^2(S+\bar{S})]^{n-1} \prod_a \mathrm{d}T^a \, \mathrm{d}\bar{T}^a$, and $[|Y^0|(S+\bar{S})]^{-2}\, \mathrm{d}Y^0 \, \mathrm{d}\bar{Y}^0 \, \mathrm{d}S \, \mathrm{d}\bar{S}$, which are separately S-duality invariant.

In (3.17) we established that $\hat{K}$ equals the black hole area on the attractor surface. For large black holes one can take the limit of large charges, keeping the dilaton field finite. Since Ω is proportional to Υ, it represents subleading terms. Therefore $\hat{K}$ yields the leading contribution to the determinants $\Delta^\pm$, so that the prefactor in the saddle-point approximation (4.8) tends to unity. Hence one recovers precisely the exponential of the macroscopic entropy. This is a gratifying result. In the saddle-point approximation the macroscopic entropy is equal to the logarithm of the microstate degeneracy up to terms that vanish in the limit of large charges. In the next section we will consider the opposite perspective and perform a similar approximation on the formula that encodes the microscopic dyonic degeneracies which yields exactly the macroscopic result encoded in (3.13) and (3.29). Hence the conjecture leading to (4.8) is clearly correct in the semiclassical approximation.

As it turns out, a similar exercise for electric (small) black holes leads to a less satisfactory situation, because the generic saddle-point expression (4.8) breaks down. This has to do with the vanishing of the determinants. In general, the vanishing of the determinant Δ^- implies that the real parts of $(Y^I, \hat{F}_I)$ are not independent coordinates and this indicates that the saddle point is not an isolated point but rather a submanifold of finite dimension at which the attractor equations will only be partially satisfied. At the saddle point $\Delta^\pm$ will vanish whenever the matrix of second derivatives of Σ at the saddle point is degenerate. This is no obstruction to a saddle-point approximation, but it implies that the general formula (4.8) is no longer applicable. For small black holes the classical contribution to the determinants

$\Delta^{\pm}$ vanishes at the attractor point, as is clear from (3.15) and (3.17). So the subleading corrections are important which tends to make approximations somewhat unreliable. Both these phenomena take place when restricting oneself to the classical terms in the measure and entropy function, and it is therefore clear that the behaviour of the integral will depend sensitively on the approximations employed.

Hence we will now perform the saddle-point approximation for 'small' (electric) black holes by following a step-by-step procedure. We assume that the magnetic attractor equations will be satisfied at the saddle-point so that we can base ourselves on (4.9). To determine the expressions for $\Delta^{\pm}$ we first recall that the charges $p^0, q_1, p^2, \ldots, p^n$ can be set to zero for the electric case. Therefore the T-moduli are equal to $T^a = -i\phi^a/\phi^0$, and thus purely imaginary. Consequently they do not contribute to the expression (3.16) for $\hat{K}$, and we obtain,

$$\hat{K}_k = 2\,(S + \bar{S})(\partial_S \Omega_k + \partial_{\bar{S}} \Omega_k)\,, \tag{4.13}$$

where k labels the particular CHL model. Substituting this result into (4.11) yields,

$$\begin{aligned}
\Delta_k^{\pm} &= \frac{(S + \bar{S})^{n+1}\,\det[-\eta_{ab}]}{(p^1)^4} \\
&\quad \times \left[\left((S + \bar{S})(\partial_S + \partial_{\bar{S}})\Omega_k \pm (S + \bar{S})^2 \partial_S \partial_{\bar{S}} \Omega_k\right)^2 - \left|(S + \bar{S})^2 D_S \partial_S \Omega_k\right|^2\right]\,,
\end{aligned} \tag{4.14}$$

which shows that the classical contribution is entirely absent and the result depends exclusively on Ω_k, defined in (3.29). Observe that here and henceforth we take $\Upsilon = -64$ and suppress the Υ-field.

We now turn to the evaluation of the inverse Laplace integral (4.9). First we write down the expression for the exponent, using (3.6) and (3.7) and rewriting ϕ^0 and ϕ^1 in terms of S and $\bar{S}$,

$$\mathcal{F}_{\rm E}(p, \phi) - q_0 \phi^0 - q_a \phi^a = -\tfrac{1}{2}(S + \bar{S})\,\phi^a \eta_{ab} \phi^b - q_a \phi^a - \frac{2q_0 p^1}{S + \bar{S}} + 4\,\Omega_k(S, \bar{S})\,. \tag{4.15}$$

We note that the above expression is not invariant under T-duality. This is due to the fact that the perturbative electric/magnetic duality basis counts p^1 as an electric and q_1 as a magnetic charge [62, 63]. Following (4.9) we consider the integral,

$$d(p^1, q_0, q_a) \propto (p^1)^2 \int \frac{dS\,d\bar{S}}{(S + \bar{S})^3} \int \prod_{a=2}^{n} d\phi^a \sqrt{\Delta_k^-(S, \bar{S})}\; e^{\pi[\mathcal{F}_{\rm E} - q_0 \phi^0 - q_a \phi^a]}\,, \tag{4.16}$$

which is not manifestly T-duality invariant. However, when performing the Gaussian integrals over ϕ^a (ignoring questions of convergence) we find

$$d(p^1, q_0, q_a) \propto (p^1)^2 \int \frac{dS\,d\bar{S}}{(S + \bar{S})^{(n+5)/2}} \sqrt{\Delta_k^-(S, \bar{S})}\; \exp\left[-\frac{\pi\,q^2}{S + \bar{S}} + 4\pi\,\Omega_k(S, \bar{S})\right]\,, \tag{4.17}$$

which is consistent with T-duality: the exponent is manifestly invariant and the explicit p^1-dependent factor cancels against a similar term in the measure factor, so that the resulting expression depends only on the T-duality invariant quantities q^2 and S. This confirms the importance of the measure factor $\sqrt{\Delta^-}$ in (4.9).

Because the real part of S becomes large for large charges, we can neglect the instanton contributions in Ω_k and use the expression (3.30). This leads to,

$$
\begin{aligned}
(S + \bar{S}) \partial_S \Omega_k &= \tfrac{1}{2}(S + \bar{S}) - \frac{k + 2}{4\pi}, \\
(S + \bar{S})^2 \partial_S \partial_{\bar{S}} \Omega_k &= \frac{k + 2}{4\pi}, \\
(S + \bar{S})^2 D_S \partial_S \Omega_k &= (S + \bar{S}) - \frac{k + 2}{4\pi}.
\end{aligned}
\tag{4.18}
$$

Substituting these results into (4.14) one obtains,

$$
\sqrt{\Delta^-(S, \bar{S})} \propto (p^1)^{-2} (S + \bar{S})^{(n+1)/2} \sqrt{\frac{k + 2}{\pi}} \sqrt{S + \bar{S} - \frac{k + 2}{2\pi}},
\tag{4.19}
$$

so that (4.17) acquires the form,

$$
d(p^1, q_0, q_a) \propto \int \frac{\mathrm{d}S \, \mathrm{d}\bar{S}}{(S + \bar{S})^{k+4}} \sqrt{S + \bar{S} - \frac{k + 2}{2\pi}} \, \exp\left[-\frac{\pi q^2}{S + \bar{S}} + 2\pi(S + \bar{S}) \right].
\tag{4.20}
$$

Let us compare this result to the result obtained in [17, 18, 20], which is also based on (4.9) but without the integration measure $\sqrt{\Delta^-}$. First of all, we note that (4.20) is manifestly invariant under T-duality, so that no ad hoc normalization factor is needed. Secondly, the above result holds irrespective of the value of n, unlike in the calculation without a measure, where one must choose the value $n = 2(k + 2) - 1$. Obviously the integral over the imaginary part of S can be performed trivially and yields an overall constant. Upon approximating the square root by $\sqrt{S + \bar{S}}$ the resulting expression (4.20) yields the following semiclassical result for the entropy,

$$
S_{\mathrm{macro}} = \log d(q^2) = 4\pi \sqrt{|q^2|/2} - \tfrac{1}{2}[(k + 2) + 1] \log |q^2|,
\tag{4.21}
$$

which disagrees with the result (5.21) of microstate counting. This is entirely due to the square root factor in the integrand of (4.20). As already noted in [20] there is a clear disagreement when the instanton contributions are retained. We may also compare to (5.26) that we shall derive later on the basis of the mixed partition function, which is also in disagreement with the above results. The situation is clearly unsatisfactory for small black holes, in sharp contrast with the situation for large black holes where there is a non-trivial agreement at the semiclassical level between the various approaches.

In order to get a better handle on the subtleties in the electric case, one may consider starting from (4.5) and integrating out the moduli fields T^a in an exact manner, rather than relying on (4.9), which is based on a saddle-point approximation. Performing the integral

over the T^a (which is Gaussian) and ignoring questions of convergence, we obtain[6]

$$d(p^1, q_0, q_a) \propto \int \frac{\mathrm{d}S \, \mathrm{d}\bar{S}}{(S + \bar{S})^2} \frac{\mathrm{d}z \, \mathrm{d}\bar{z} \, |z|^{n+1}}{(1 - z - \bar{z})^{(n-1)/2}} \, e^{\pi \Sigma_{\mathrm{eff}}}$$

$$\times \left\{ \frac{n(n-1)(z+\bar{z})^2}{8\pi^2 |z|^4 (1 - z - \bar{z})^2} - \frac{|(S+\bar{S})^2 D_S \partial_S \Omega_k|^2}{|z|^4} \right.$$

$$\left. + \left[\frac{(n-1)(z+\bar{z})}{4\pi |z|^2 (1 - z - \bar{z})} - \frac{(S+\bar{S})\partial_S \Omega_k}{z^2} - \frac{(S+\bar{S})\partial_{\bar{S}} \Omega_k}{\bar{z}^2} + \frac{(S+\bar{S})^2 \partial_S \bar{\partial}_{\bar{S}} \Omega_k}{|z|^2} \right]^2 \right\},$$

$$(4.22)$$

where Σ_{eff} is given by

$$\Sigma_{\mathrm{eff}} = -q^2 \frac{z + \bar{z}}{2(S + \bar{S})} + 4\Omega_k - \frac{2(\bar{z}-1)}{z}(S + \bar{S})\partial_S \Omega_k - \frac{2(z-1)}{\bar{z}}(S + \bar{S})\partial_{\bar{S}} \Omega_k \,, \qquad (4.23)$$

and where z is given by the S-duality invariant variable,

$$z = \frac{Y^0 (S + \bar{S})}{\bar{P}(\bar{S})} \,, \qquad (4.24)$$

where $P(S)$, defined in (3.9), equals p^1 in the case at hand. We observe that the integral (4.22) is far more complicated than the expression (4.17) resulting from (4.9). In particular, we observe that the solution for Y^a induced by integrating out the T^a reads

$$Y^a = -\frac{\eta^{ab} q_b}{2(S + \bar{S})} \, z \,. \qquad (4.25)$$

Comparing with (3.8) shows that this only coincides with the attractor value for Y^a when $z = 1$. Actually, the latter equation is itself one of the attractor equations, as is clear from the first equation in (3.8) . For this particular value of z, $\pi \Sigma_{\mathrm{eff}}$ coincides with the exponent in (4.17). Clearly, in order to better exhibit the difference between (4.22) and (4.17), it is crucial to perform the integral over the variable z. This, however, is a complicated integral. On the other hand, evaluating (4.22) in saddle-point approximation, neglecting as before the instanton contributions, gives again the result (4.21) while the saddle point is still located at $z = 1$.

5 More on heterotic black holes in $N = 4$ compactifications

In this section we will use the expressions for microscopic black hole degeneracies to make contact with the macroscopic results described in the previous sections. Examples of these microscopic degeneracies are provided by heterotic string theory compactified on a six-torus and by the class of heterotic CHL models [53]. All these models have $N = 4$ supersymmetry. A conjecture for the associated microstate degeneracy has been put forward sometime ago in [60] for the case of toroidally compactified heterotic string theory, and more recently in

---

[6]There exists an analogous version of this formula for the case of dyonic black holes.

[55] for the case of CHL models. In the toroidal case, the degeneracy is based on the unique automorphic form Φ_{10} of weight 10 under the genus two modular group $\mathrm{Sp}(2, \mathbb{Z})$. This proposal has recently received further support in [61]. For the CHL models, the degeneracy is based on the modular form Φ_k of weight k under a subgroup of the genus-two modular group $\mathrm{Sp}(2, \mathbb{Z})$. It should be noted that the second proposal is only applicable for states carrying electric charges arising from the twisted sector [55].

It can be shown [43, 55] that, for large charges, the asymptotic growth of the degeneracy of 1/4-BPS dyons in the models discussed above precisely matches the macroscopic entropy of dyonic black holes given in (3.13), with the dilaton S determined in terms of the charges through (3.15). This is reviewed in the next subsection.

5.1 Asymptotic growth

The degeneracy of 1/4-BPS dyons in the class of models discussed above, is captured by automorphic forms $\Phi_k(\rho, \sigma, \upsilon)$ of weight k under $\mathrm{Sp}(2, \mathbb{Z})$ or an appropriate subgroup thereof [60, 55]. The three modular parameters, ρ, σ, υ, parametrize the period matrix of an auxiliary genus-two Riemann surface which takes the form of a complex, symmetric, two-by-two matrix. The case $k = 10$ corresponds to Φ_{10}, which is the relevant modular form for the case of toroidal compactifications [60]. The microscopic degeneracy of 1/4-BPS dyons in a given model takes the form of an integral over an appropriate 3-cycle,

$$d_k(p, q) = \oint \mathrm{d}\rho \, \mathrm{d}\sigma \, \mathrm{d}\upsilon \, \frac{\mathrm{e}^{\mathrm{i}\pi[\rho \, p^2 + \sigma \, q^2 + (2\upsilon - 1) \, p \cdot q]}}{\Phi_k(\rho, \sigma, \upsilon)} \ , \tag{5.1}$$

where we have included a shift of υ, following [58]. It is important to note that the charges are in general integer, with the exception of q_1 which equals a multiple of N, and p^1 which is fractional and quantized in units of $1/N$. Consequently $p^2/2$ and $p \cdot q$ are quantized in integer units, whereas $q^2/2$ is quantized in units of $1/N$. The inverse of the modular form Φ_k takes the form of a Fourier sum with integer powers of $\exp[2\pi \mathrm{i}\rho]$ and $\exp[2\pi \mathrm{i}\upsilon]$ and fractional powers of $\exp[2\pi \mathrm{i}\sigma]$ which are multiples of $1/N$. The 3-cycle is then defined by choosing integration contours where the real parts of ρ and υ take values in the interval $(0, 1)$ and the real part of σ takes values in the interval $(0, N)$. The precise definition of Φ_k is subtle and we refer to [55] for further details.

The formula (5.1) is invariant under both S-duality, which is a subgroup of the full modular group, and T-duality. Target-space duality invariance is manifest, as the integrand only involves the three T-duality invariant combination of the charges given in (3.14). To exhibit the invariance under S-duality, one makes use of the transformation properties of the charges as well as of the integration variables ρ, σ, υ. Since the result depends on the choice of an integration contour, S-duality invariance is only formal at this point.

The function Φ_k has zeros which induce corresponding poles in the integrand whose residues will yield the microscopic degeneracy. Since Φ_k has zeros in the interior of the Siegel half-space in addition to the zeros at the cusps, the value of the integral (5.1) depends on the choice of the integration 3-cycle. It is possible to determine the poles of Φ_k^{-1} which are

responsible for the leading and subleading contributions to $d_k(p, q)$, as was first shown in [43] for $k = 10$, and generalized recently to other values of k in [55]. Below we briefly summarize this result.

When performing an asymptotic evaluation of the integral (5.1), one must specify which limit in the charges is taken. 'Large' black holes correspond to a limit where both electric and magnetic charges are taken to be large. More precisely, one takes $q^2 p^2 - (p \cdot q)^2 \gg 1$, and $q^2 + p^2$ must be large and negative. This implies that the classical entropy and area of the corresponding black holes are finite. Under a uniform scaling of the charges the dilaton will then remain finite; to ensure that it is nevertheless large one must assume that $|p^2|$ is sufficiently small as compared to $\sqrt{q^2 p^2 - (p \cdot q)^2}$. In this way one can recover the nonperturbative string corrections, as was stressed in [43].

The leading behaviour of the dyonic degeneracy is associated with the rational quadratic divisor $\mathcal{D} = \upsilon + \rho\sigma - \upsilon^2 = 0$ of Φ_k, near which Φ_k takes the form,

$$\frac{1}{\Phi_k(\rho, \sigma, \upsilon)} \approx \frac{1}{\mathcal{D}^2} \frac{1}{\sigma^{-(k+2)} f^{(k)}(\gamma') f^{(k)}(\sigma')} + \mathcal{O}(\mathcal{D}^0) \,, \tag{5.2}$$

where

$$\gamma' = \frac{\rho\sigma - \upsilon^2}{\sigma} \quad, \quad \sigma' = \frac{\rho\sigma - (\upsilon - 1)^2}{\sigma} \,. \tag{5.3}$$

The cusp forms $f^{(k)}$ and their transformation rules have been defined in (3.26) and (3.27), respectively. Here we note that [43] and [55] differ from one another in the way the forms Φ_k are expanded and in the expansion variables used. This, for instance, results in different definitions of σ', which do, however, agree on the divisor. Here we follow [55].

Clearly, Φ_k has double zeros at $\upsilon_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4\rho\sigma}$ on the divisor. The evaluation of the integral (5.1) proceeds by first evaluating the contour integral for υ around either one of the poles $\upsilon_\pm$, and subsequently evaluating the two remaining integrals over ρ and σ in saddle-point approximation. The saddle-point values of ρ, σ, and hence of $\upsilon_\pm$, can be parametrized by

$$\rho = \frac{\mathrm{i}|S|^2}{S + \bar{S}} \,, \qquad \sigma = \frac{\mathrm{i}}{S + \bar{S}} \,, \qquad \upsilon_\pm = \frac{S}{S + \bar{S}} \,. \tag{5.4}$$

As argued in [43], these values describe the unique solution to the saddle-point equations for which the state degeneracy $d_k(p, q)$ takes a real value. The resulting expression for $\log d_k(p, q)$ precisely equals (3.13), with S given by the dilaton and expressed in terms of the charges through the attractor equation (3.15). The result is valid up to a constant and up to terms that are suppressed by inverse powers of the charges. Other divisors are expected to give rise to exponentially suppressed corrections to the microscopic entropy $S_{\text{micro}} = \log d_k(p, q)$.

The asymptotic degeneracy can also be compared with the expression (4.8). However, we already argued that for large black holes, the ratio of the two determinants equals one, up to subleading terms that are inversely proportional to the charges. Therefore, to this order of accuracy, the asymptotic degeneracy computed from (5.1) is in precise agreement with (4.8), and hence correctly reproduced by the proposal (4.5).

5.2 The mixed partition function

A more refined test of the proposal (4.5) consists in checking whether the mixed partition sum $Z_k(p, \phi)$ associated with the microstate degeneracies $d_k(p, q)$ and defined by the first equation of (4.10), agrees with the right-hand side of that same equation. The latter was derived from (4.5) by a saddle-point integration over the imaginary part of the Y^I. As discussed in section 4, the right-hand side of (4.10) may require an explicit periodicity sum over discrete imaginary shifts of the ϕ. Since the right-hand side of (4.10) results from a saddle-point evaluation, we expect to find perturbative as well as non-perturbative corrections to it. Both these features will show up in the examples discussed below.

In the following, we will first compute the mixed partition function $Z_k(p, \phi)$ for $N = 4$ dyons (corresponding to large black holes) in the class of $N = 4$ models discussed above. For toroidally compactified heterotic string theory, this mixed partition function was recently computed in [58] for the case when $p^0 = 0$. A generalization to the case $p^0 \neq 0$ was reported in [21]. Here the same techniques are used to compute the mixed partition function for CHL models. Next, we compute the (reduced) partition function for electrically charged 1/2-BPS states (which correspond to small black holes) in the same class of models.

5.2.1 Large (dyonic) black holes

We start by noting that the microstate degeneracies must be consistent with T-duality, so that the $d_k(p, q)$ can be expressed in terms of the three invariants $Q \equiv q^2$, $P \equiv p^2$ and $R \equiv p \cdot q$. Therefore we replace the sums over q_0 and q_1 in (4.10) by sums over the charges Q and P, related by the identities,[7]

$$q_0 = \frac{1}{2p^1}(Q + \tfrac{1}{2}q_a \eta^{ab} q_b), \qquad q_1 = -\frac{1}{2p^0}(P + 2p^a \eta_{ab} p^b). \tag{5.5}$$

However, only those values of Q and P are admissible that lead to integer-valued charges q_0 and charges q_1 that are a multiple of N, also taking into account that Q is quantized in units of $2/N$ and that P is even. These restrictions can be implemented by inserting the series $L^{-1}\sum_{l=0}^{L-1} \exp[2\pi i\, l\, K/L]$, where K and L are integers (with L positive), which projects onto all integer values for K/L. The use of this formula leads to the following expression,

$$Z_k(p, \phi) \;=\; \frac{1}{N^2 p^0 p^1} \sum_{\substack{\phi^0 \to \phi^0 + 2i\, l^0 \\ \phi^1 \to \phi^1 + 2i\, l^1/N}} \sum_{q_a, Q, P} d_k(Q, P, R)$$

$$\times \exp\left[\frac{\pi \phi^0}{2p^1}(Q + \tfrac{1}{2}q_a \eta^{ab} q_b) - \frac{\pi \phi^1}{2p^0}(P + 2p^a \eta_{ab} p^b) + \pi q_a \phi^a\right], \tag{5.6}$$

with R given by

$$R = \frac{p^0}{2p^1}(Q + \tfrac{1}{2}q_a \eta^{ab} q_b) + \frac{p^1}{2p^0}(P + 2p^a \eta_{ab} p^b) + q_a p^a. \tag{5.7}$$

---

[7]Observe that we will be assuming that both p^0 and p^1 are non-vanishing. When $p^0 = 0$, as was the case in [58], the unrestricted sums can be replaced by sums over Q and R.

In (5.6) the summation over imaginary shifts of ϕ^0 and ϕ^1 is implemented by first replacing $\phi^0 \to \phi^0 + 2\mathrm{i}l^0$ and $\phi^1 \to \phi^1 + 2\mathrm{i}l^1/N$ in each summand, and subsequently summing over the integers $l^0 = 0, \ldots, Np^1 - 1$ and $l^1 = 0, \ldots, Np^0 - 1$. The sums over $l^{0,1}$ enforce that only those summands, for which $(Q + \frac{1}{2}q_a\eta^{ab}q_b)/2p^1$ is an integer and $(P + 2p^a\eta_{ab}p^b)/2p^0$ is a multiple of N, give a non-vanishing contribution to $Z_k(p, \phi)$.

Next, we perform the sums over Q and P without any restriction, using (5.1) and taking into account that $NQ/2$ and $P/2$ are integer valued. Provided we make a suitable choice for the integration contours for σ and ρ, both sums can be rewritten as sums of delta-functions, which imply that σ and ρ are equal to $\sigma(v)$ and $\rho(v)$, up to certain integers, where

$$
\begin{aligned}
\sigma(v) &= -\frac{\phi^0}{2\mathrm{i}p^1} - (2v - 1)\frac{p^0}{2p^1}\,, \\
\rho(v) &= \frac{\phi^1}{2\mathrm{i}p^0} - (2v - 1)\frac{p^1}{2p^0}\,.
\end{aligned}
\tag{5.8}
$$

The required choice for the integration contours implies $\mathrm{Im}\,\sigma = \mathrm{Im}\,\sigma(v)$ and $\mathrm{Im}\,\rho = \mathrm{Im}\,\rho(v)$, for given v. The sums of delta-functions then take the form $\sum_{n\in\mathbb{Z}} N\,\delta(\mathrm{Re}\,\sigma - \mathrm{Re}\,\sigma(v) - nN)$ and $\sum_{m\in\mathbb{Z}} \delta(\mathrm{Re}\,\rho - \mathrm{Re}\,\rho(v) - m)$, respectively. Note that the shifts in the arguments of the delta-functions are precisely generated by additional shifts of ϕ^0 and ϕ^1,

$$
\phi^0 \to \phi^0 + 2\mathrm{i}p^1 N\,n\,, \qquad \phi^1 \to \phi^1 + 2\mathrm{i}p^0\,m\,.
\tag{5.9}
$$

However, the resulting integrals do not depend on these shifts as they turn out to be periodic under (5.9). Therefore only one of the delta-function contributes, so that the integrations over σ and ρ result in

$$
Z_k(p, \phi) = \frac{1}{N\,p^0 p^1} \sum_{\substack{\phi^0 \to \phi^0 + 2\mathrm{i}\,l^0 \\ \phi^1 \to \phi^1 + 2\mathrm{i}\,l^1/N}} \sum_{q_a} \int \mathrm{d}v\, \frac{1}{\Phi_k(\rho(v), \sigma(v), v)}
$$
$$
\times \exp\left(-\mathrm{i}\pi\left[\tfrac{1}{2}\sigma(v)\,q_a\eta^{ab}q_b + 2\rho(v)\,p^a\eta_{ab}p^b + \mathrm{i}q_a\left(\phi^a + \mathrm{i}(2v - 1)p^a\right)\right]\right).
\tag{5.10}
$$

Since the integrand is invariant under the shifts (5.9), the explicit sum over shifts with $l^0 = 0, \ldots, Np^1 - 1$ and $l^1 = 0, \ldots, Np^0 - 1$ ensures that the partition function (5.10) is invariant under shifts $\phi^a \to \phi^a + 2\mathrm{i}$, as well as under shifts $\phi^0 \to \phi^0 + 2\mathrm{i}$ and $\phi^1 \to \phi^1 + 2\mathrm{i}/N$.

Subsequently we perform a formal Poisson resummation of the charges q_a, i.e., we ignore the fact that η_{ab} is not positive definite. We obtain (up to an overall numerical constant),

$$
\begin{aligned}
Z_k(p, \phi) &= \frac{\sqrt{\det[-\eta_{ab}]}}{N\,p^0 p^1} \sum_{\text{shifts}} \int \mathrm{d}v\, \frac{1}{\sigma(v)^{(n-1)/2}\,\Phi_k(\rho(v), \sigma(v), v)} \\
&\times \exp\left(-\mathrm{i}\pi\left[2p^a\eta_{ab}p^b\,\rho(v) + \frac{(\phi^a + \mathrm{i}(2v - 1)p^a)\,\eta_{ab}\,(\phi^b + \mathrm{i}(2v - 1)p^b)}{2\sigma(v)}\right]\right),
\end{aligned}
\tag{5.11}
$$

where here and henceforth the sum over shifts denotes the infinite sum over shifts $\phi^a \to \phi^a + 2\mathrm{i}$, together with the finite sums over shifts in ϕ^0 and ϕ^1. This result is completely in line with what was discussed below (4.3).

Now we perform the contour integral over v. This integration picks up the contributions from the residues at the various poles of the integrand. The leading contribution to this sum of residues stems from the zeros of Φ_k. For large magnetic charges p and large scalars ϕ, the leading contribution to the mixed partition function $Z_k(p, \phi)$ is expected to be associated with the rational quadratic divisor $\mathcal{D} = v + \rho\sigma - v^2 = 0$ of Φ_k, near which Φ_k takes the form (5.2) [43, 55]. This is the divisor responsible for the leading contribution to the entropy [60, 43, 55], and hence it is natural to expect that this divisor also gives rise to the leading contribution to the free energy, and therefore to the mixed partition function. Then, other poles of the integrand in (5.11) give rise to exponentially suppressed contributions.

Inserting $\rho(v)$ and $\sigma(v)$ into $\mathcal{D}$ yields

$$\mathcal{D} = 2(v - v_*) \frac{\phi^0 p^1 - \phi^1 p^0}{4ip^0 p^1} , \tag{5.12}$$

with v_* given by

$$2v_* = 1 - i \frac{\phi^0 \phi^1 + p^1 p^0}{\phi^0 p^1 - \phi^1 p^0} . \tag{5.13}$$

We observe that the quadratic piece in v has canceled, and that $\mathcal{D}$ has therefore a simple zero. Performing the contour integral over v then yields (again, up to an overall numerical constant),

$$Z_k(p, \phi) = \frac{p^0 p^1 \sqrt{\det[-\eta_{ab}]}}{N} \sum_{\text{shifts}} \frac{1}{(\phi^0 p^1 - \phi^1 p^0)^2}$$
$$\times \frac{d}{dv} \left[\frac{\exp\left(-i\pi \left[2p^a \eta_{ab} p^b \rho(v) + \frac{(\phi^a + i(2v-1)p^a)\, \eta_{ab}\, (\phi^b + i(2v-1)p^b)}{2\sigma(v)}\right]\right)}{\sigma(v)^{(n-2k-5)/2}\, f^{(k)}(\gamma'(v))\, f^{(k)}(\sigma'(v))} \right]_{v=v_*} , \tag{5.14}$$

where we made use of (5.2) and we discarded the exponentially suppressed contributions originating from other possible poles of the integrand.

Using (5.8) we can determine the following expressions for the derivatives with respect to v on the divisor,

$$\begin{aligned}
\frac{d\rho(v)}{dv}\bigg|_{v=v_*} &= -\frac{p^1}{p^0} , & \frac{d\gamma'(v)}{dv}\bigg|_{v=v_*} &= -\frac{[p^1\sigma + p^0 v]^2}{p^0 p^1\, \sigma^2} , \\
\frac{d\sigma(v)}{dv}\bigg|_{v=v_*} &= -\frac{p^0}{p^1} , & \frac{d\sigma'(v)}{dv}\bigg|_{v=v_*} &= -\frac{[p^1\sigma + p^0(v-1)]^2}{p^0 p^1\, \sigma^2} ,
\end{aligned} \tag{5.15}$$

where v, ρ and σ on the right-hand side refer to the values of these variables on the divisor,

i.e., v_*, $\rho_* = \rho(v_*)$ and $\sigma_* = \sigma(v_*)$. Inserting the above expressions into (5.14), we obtain,

$$
Z_k(p, \phi) = \frac{\sqrt{\det[-\eta_{ab}]}}{N} \sum_{\text{shifts}} \frac{\sigma^{-(n+3)/2}}{(\phi^0 p^1 - \phi^1 p^0)^2}
$$

$$
\times \exp\Big(-\frac{i\pi}{2\sigma} \Big[\phi^a \eta_{ab} \phi^b - p^a \eta_{ab} p^b + 2i\, p^a \eta_{ab} \phi^b (2v - 1) \Big]
$$

$$
- \ln f^{(k)}(-v/\sigma) - \ln f^{(k)}((v-1)/\sigma) + (k+2) \ln \sigma \Big)
$$

$$
\times \Big[-\tfrac{1}{2} i\pi (p^a \phi^0 - p^0 \phi^a) \eta_{ab} (p^b \phi^0 - p^0 \phi^b) + \tfrac{1}{2}(n - 2k - 5)(p^0)^2 \sigma
$$

$$
+ [p^1 \sigma + p^0 v]^2 \big[\ln f^{(k)}(-v/\sigma) \big]' + [p^1 \sigma + p^0(v - 1)]^2 \big[\ln f^{(k)}((v-1)/\sigma) \big]' \Big].
$$

(5.16)

To make contact with the macroscopic expressions, we define S and $\bar{S}$ in terms of ϕ^0 and ϕ^1, according to (3.7). Substituting the corresponding expressions into (5.8) and (5.13), we recover precisely the expressions for the divisor values of ρ, σ, v in terms of S and $\bar{S}$ that were shown in (5.4). Likewise we use $T^a = (-i\phi^a + p^a)/(\phi^0 + ip^0)$ and $\bar{T}^a = (i\phi^a + p^a)/(\phi^0 - ip^0)$. Note that under the periodicity shifts, $S, \bar{S}, T^a, \bar{T}^a$ should be treated as functions of ϕ^I. After being subjected to such a shift, S and $\bar{S}$, and T^a and $\bar{T}^a$, respectively, are no longer related by complex conjugation.

We now note that (5.16) takes the same form as (4.10). The exponential factor in (5.16) coincides precisely with $\exp[\pi \mathcal{F}_{\mathrm{E}}(p, \phi)]$ after substituting (5.4), so that the prefactor $\sqrt{\Delta^-}$ should be identified with the remaining terms. Hence we obtain (up to an overall numerical constant),

$$
\sqrt{\Delta^-} = \frac{(S + \bar{S})^{(n-3)/2} \sqrt{\det[-\eta_{ab}]}}{2\, |Y^0|^2} \left(\hat{K} + 4(S + \bar{S})^2 \partial_S \partial_{\bar{S}} \Omega + \frac{(n-1)}{4\pi} \frac{(Y^0 - \bar{Y}^0)^2}{|Y^0|^2} \right), \quad (5.17)
$$

where we also used that $Y^0 = (\phi^0 + ip^0)/2$. The above results for the mixed partition function of $N = 4$ dyons (with generic charges) in CHL models is exact, up to exponentially suppressed corrections. When setting $p^0 = 0$ the resulting expression for the toroidal case ($k = 10$) agrees with the one found in [58], up to a subtlety involving the periodicity sums. The expression (5.17) is consistent with our previous result (4.11) in the limit of large charges. In that limit the term proportional to $\hat{K}$ dominates, as we explained in section 4. Recall, however, that $\sqrt{\Delta^-(p, \phi)}$ enters into (4.10) in the context of a saddle-point approximation, which is expected to be subject to further perturbative and non-perturbative corrections.

There is, however, an issue with regard to the number of moduli, which depends on the integer n. In the context of the above calculation, $n+1$ equals the rank of the gauge group of the corresponding CHL model. On the other hand, in the context of $N = 2$ supersymmetry n defines the number of matter vector supermultiplets. In this case the rank of the gauge group is still also equal to $n + 1$. However, the value taken for n in the case of $N = 4$ supersymmetry differs from the $N = 2$ value by four. The difference is related to the six graviphotons of pure $N = 4$ supergravity, whose $N = 2$ decomposition is as follows. One graviphoton belongs to the $N = 2$ graviton multiplet, another graviphoton belongs to an

$N = 2$ vector multiplet, whereas the four remaining graviphotons belong to two $N = 2$ gravitino supermultiplets. When the description of the $N = 4$ supersymmetric black holes is based on $N = 2$ supergravity, the above results seem to indicate that the charges (and the corresponding electrostatic potentials ϕ) associated with the extra gravitini should be taken into account. This question is particularly pressing for the small (electric) $N = 4$ black holes, where saddle-point approximations are more cumbersome. For that reason we will briefly reconsider the mixed partition function for the case of small black holes in the next subsection.

5.2.2 Small (electric) black holes

Here, we compute the partition function for electrically charged 1/2-BPS states in CHL models. These are states with vanishing charges q_1, p^0 and p^a. We therefore consider the reduced partition sum,

$$Z_{\mathrm{R}}(p, \phi) = \sum_{q_0, q_a} d(q^2)\, \mathrm{e}^{\pi[q_0\phi^0 + q_a\phi^a]} \,. \tag{5.18}$$

As in the previous subsection, we replace the sum over the charges q_0 by a sum over the charges $Q \equiv q^2$, where we recall that Q is quantized in units of $2/N$ [55]. Following the same step as in the derivation of (5.6), this results in

$$Z_{\mathrm{R}}(p, \phi) = \frac{1}{N\, p^1} \sum_{\phi^0 \to \phi^0 + 2\mathrm{i}l^0} \sum_{q_a, Q} d(Q)\, \exp\left[\frac{\pi\phi^0}{2p^1}(Q + \tfrac{1}{2}q_a\eta^{ab}q_b) + \pi q_a\phi^a\right] \,. \tag{5.19}$$

Here, the integers l^0 run over $l^0 = 0, \ldots, Np^1 - 1$.

Next, we perform the sum over Q by using the integral expression for the electric degeneracies [55],

$$d(Q) = \oint \mathrm{d}\sigma\, \sigma^{k+2}\, \frac{\mathrm{e}^{\mathrm{i}\pi\sigma Q}}{f^{(k)}(-1/\sigma)} \,, \tag{5.20}$$

where σ runs in the strip $\sigma \sim \sigma + N$. Observe that (5.20) has the asymptotic expansion,

$$\log d(Q) = 4\pi\, \sqrt{\tfrac{1}{2}|Q|} - \tfrac{1}{2}[(k + 2) + \tfrac{3}{2}]\, \log|Q| \,. \tag{5.21}$$

By making a suitable choice for the integration contour of σ, the sum over Q can be rewritten as a sum over delta-functions, $\sum_{n \in \mathbb{Z}} N\, \delta(\mathrm{Re}\,\sigma - \mathrm{Re}\,\sigma_* - nN)$, where $\sigma_* = -\phi^0/(2\mathrm{i}p^1)$. The sum over n is generated by imaginary shifts of ϕ^0 according to

$$\phi^0 \to \phi^0 + 2\mathrm{i}p^1 N\, n \,, \tag{5.22}$$

just as before. However, the resulting integral does not depend on these shifts as it is periodic under (5.22). Therefore only one of the delta-function contributes, so that the integration over σ results in

$$Z_{\mathrm{R}}(p, \phi) = \frac{1}{p^1} \sum_{\phi^0 \to \phi^0 + 2\mathrm{i}l^0} \frac{\sigma_*^{k+2}}{f^{(k)}(-1/\sigma_*)} \sum_{q_a} \exp\left(\mathrm{i}\pi\left[-\tfrac{1}{2}\sigma_*\, q_a\eta^{ab}q_b - \mathrm{i}q_a\, \phi^a\right]\right) \,. \tag{5.23}$$

Since (5.23) is invariant under the shifts (5.22), the explicit sum over shifts with $l^0 = 0, \ldots, Np^1 - 1$ ensures that the reduced partition function (5.23) is invariant under shifts $\phi^a \to \phi^a + 2i$ as well as under shifts $\phi^0 \to \phi^0 + 2i$.

The next step is to perform a Poisson resummation of the charges q_a, ignoring, as before, that η_{ab} is not positive definite. This yields (up to an overall numerical constant),

$$Z_{\mathrm{R}}(p, \phi) = \frac{\sqrt{\det[-\eta_{ab}]}}{p^1} \sum_{\phi^{0,a} \to \phi^{0,a} + 2il^{0,a}} \frac{\sigma_*^{-(n-1)/2+k+2}}{f^{(k)}(-1/\sigma_*)} \exp\left(-i\pi \frac{\phi^a \, \eta_{ab} \, \phi^b}{2\sigma_*}\right) , \qquad (5.24)$$

where $l^a \in \mathbb{Z}$.

Now we recast (5.24) in terms of the scalar field S given in (3.7). Because $p^0 = 0$ in the electric case, $S = (p^1 - i\phi^1)/\phi^0$ so that $\sigma_* = i/(S + \bar{S})$, precisely as in (5.4). Clearly, the result (5.24) can now be factorized as follows (again up to an overall numerical constant),

$$Z_{\mathrm{R}}(p, \phi) = \sum_{\text{shifts}} \sqrt{\tilde{\Delta}^-(p, \phi)} \, e^{\pi \, \tilde{\mathcal{F}}_{\mathrm{E}}(p,\phi)} , \qquad (5.25)$$

where

$$\begin{aligned}
\tilde{\mathcal{F}}_{\mathrm{E}}(p, \phi) &= -\tfrac{1}{2}(S + \bar{S})\phi^a \eta_{ab}\phi^b - \frac{(k+2)}{\pi} \log(S + \bar{S}) - \frac{1}{\pi} \log f^{(k)}(S + \bar{S}) , \\
\tilde{\Delta}^-(p, \phi) &= \frac{\det[-\eta_{ab}]}{(p^1)^2} (S + \bar{S})^{n-1} .
\end{aligned} \qquad (5.26)$$

Although (5.25) is of the same form as the right-hand side of (4.10), the quantities $\tilde{\mathcal{F}}_{\mathrm{E}}(p, \phi)$ and $\tilde{\Delta}^-(p, \phi)$ do not at all agree with (3.6) and (4.14). One of the most conspicuous features is the fact that the partition function does not depend on ϕ^1, which is proportional to the imaginary part of S. This is the result of the fact that for the electric black hole we took $q_1 = 0$. This is undoubtedly related to the electric/magnetic duality basis that has to be used here [62, 63]. When suppressing the instanton corrections, both (3.6) and (4.14) also become functions of the real part of S. In that case, $\tilde{\mathcal{F}}_{\mathrm{E}}(p, \phi)$ does coincide with (3.6), but there is no way to reconcile $\tilde{\Delta}^-(p, \phi)$ with (4.19).

Acknowledgements

References

[1] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy", *Phys. Lett.* **B379** (1996) 99–104, `hep-th/9601029`.

[2] J.M. Maldacena, A. Strominger and E. Witten, "Black hole entropy in M-theory", *JHEP* **12** (1997) 002, `hep-th/9711053`.

[3] C. Vafa, "Black holes and Calabi-Yau threefolds", *Adv. Theor. Math. Phys.* **2** (1998) 207–218, `hep-th/9711067`.

[4] G.L. Cardoso, B. de Wit and T. Mohaupt, "Corrections to macroscopic supersymmetric black-hole entropy", *Phys. Lett.* **B451** (1999) 309–316, `hep-th/9812082`.

[5] R.M. Wald, "Black hole entropy entropy is Noether charge", *Phys. Rev.* **D48** (1993) 3427, `gr-qc/9307038`; V. Iyer and R.M. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy", *Phys. Rev.* **D50** (1994) 846, `gr-qc/9403028`; T. Jacobson, G. Kang and R.C. Myers, "Black hole entropy in higher curvature gravity", *Phys. Rev.* **D49** (1994) 6587, `gr-qc/9312023`.

[6] G.L. Cardoso, B. de Wit, J. Käppeli and T. Mohaupt, "Stationary BPS solutions in $N = 2$ supergravity with R^2-interactions", *JHEP* **12** (2000) 019, `hep-th/0009234`.

[7] H. Ooguri, A. Strominger and C. Vafa, "Black hole attractors and the topological string", *Phys. Rev.* **D70** (2004) 106007, `hep-th/0405146`.

[8] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, "Holomorphic anomalies in topological field theories", *Nucl. Phys.* **B405** (1993) 279-304, `hep-th/9302103`; "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes", *Commun. Math. Phys.* **165** (1994) 311-428, `hep-th/9309140`.

[9] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, "Topological amplitudes in string theory", *Nucl. Phys.* **B413** (1994) 162, `hep-th/9307158`.

[10] C. Vafa, "Two dimensional Yang-Mills, black holes and topological strings," `hep-th/0406058`.

[11] M. Aganagic, H. Ooguri, N. Saulina and C. Vafa, "Black holes, q-deformed 2d Yang-Mills, and non-perturbative topological strings," *Nucl. Phys.* **B715** (2005) 304, `hep-th/0411280`.

[12] N. Caporaso, M. Cirafici, L. Griguolo, S. Pasquetti, D. Seminara and R.J. Szabo, "Black-holes, topological strings and large N phase transitions", `hep-th/0512213`.

[13] M. Aganagic, D. Jafferis and N. Saulina, "Branes, Black Holes and Topological Strings on Toric Calabi-Yau Manifolds", `hep-th/0512245`.

[14] A.A. Gerasimov and S.L. Shatashvili, "Towards integrability of topological strings. I: Three-forms on Calabi-Yau manifolds". *JHEP* **0411** (2004) 074, `hep-th/0409238`.

[15] E. Verlinde, "Attractors and the holomorphic anomaly", `hep-th/0412139`.

[16] H. Ooguri, C. Vafa and E. P. Verlinde, "Hartle-Hawking wave-function for flux compactifications", `hep-th/0502211`.

[17] A. Dabholkar, "Exact counting of black hole microstates", *Phys. Rev. Lett.* **94** (2005) 241301, `hep-th/0409148`.

[18] A. Dabholkar, F. Denef, G.W. Moore and B. Pioline, "Exact and asymptotic degeneracies of small black holes", *JHEP* **0508** (2005) 021, `hep-th/0502157`.

[19] A. Sen, "Black holes and the spectrum of half-BPS States in $N = 4$ supersymmetric string theory", `hep-th/0504005`.

[20] A. Dabholkar, F. Denef, G.W. Moore and B. Pioline, "Precision counting of small black holes", *JHEP* **0510** (2005) 096, `hep-th/0507014`.

[21] J. Käppeli, "Partition function of dyonic black holes in $N = 4$ string theory", `hep-th/0511221`.

[22] B. de Wit, "Supersymmetric black holes", *Fortschr. Phys.* **54** (2006) 183, `hep-th/0511261`.

[23] T. Mohaupt, "Strings, higher curvature corrections and black holes", `hep-th/0512048`.

[24] S. Ferrara, R. Kallosh and A. Strominger, "$N = 2$ extremal black holes", *Phys. Rev.* **D52** (1995) 5412–5416, `hep-th/9508072`.

[25] A. Strominger, "Macroscopic entropy of $N = 2$ extremal black holes", *Phys. Lett.* **B383** (1996) 39–43, `hep-th/9602111`.

[26] S. Ferrara and R. Kallosh, "Supersymmetry and attractors", *Phys. Rev.* **D54** (1996) 1514–1524, `hep-th/9602136`.

[27] G.W. Gibbons, unpublished notes.

[28] S. Ferrara, G.W. Gibbons and R. Kallosh, "Black holes and critical points in moduli space", *Nucl. Phys.* **B500** (1997) 75-93, `hep-th/9702103`.

[29] A. Sen, "Black hole entropy function and the attractor mechanism in higher derivative gravity", *JHEP* **0509** (2005) 038, `hep-th/0506177`.

[30] K. Goldstein, N. Iizuka, R.P. Jena and S.P. Trivedi, "Non-supersymmetric attractors", `hep-th/0507096`.

[31] A. Sen, "Entropy function for heterotic black holes", `hep-th/0508042`.

[32] R. Kallosh, "New attractors", `hep-th/0510024`.

[33] P.K. Tripathy and S.P. Trivedi, "Non-supersymmetric attractors in string theory", `hep-th/0511117`.

[34] A. Giryavets, "New attractors and area codes", `hep-th/0511215`.

[35] P. Prester, "Lovelock type gravity and small black holes in heterotic string theory", `hep-th/0511306`.

[36] K. Goldstein, R.P. Jena, G. Mandal and S.P. Trivedi, "A C-function for non-supersymmetric attractors", `hep-th/0512138`.

[37] M. Alishahiha and H. Ebrahim, "Non-Supersymmetric Attractors and Entropy Function", `hep-th/0601016`.

[38] K. Behrndt, G.L. Cardoso, B. de Wit, R. Kallosh, D. Lüst and T. Mohaupt, "Classical and quantum $N = 2$ supersymmetric black holes", *Nucl. Phys.* **B488** (1997) 236-260, `hep-th/9610105`.

[39] B. de Wit, "$N = 2$ Electric-magnetic duality in a chiral background", *Nucl. Phys. Proc. Suppl.* **49** (1996) 191-200, `hep-th/9602060`; "$N = 2$ symplectic reparametrizations in a chiral background", *Fortschr. Phys.* **44** (1996) 529-538, `hep-th/9603191`.

[40] A. Sen, "Black hole solutions in heterotic string theory on a torus", *Nucl. Phys.* **B440** (1995) 421, `hep-th/9411187`.

[41] A. Sen, "Extremal black holes and elementary string states", *Mod. Phys. Lett.* **A10** (1995) 2081, `hep-th/9504147`.

[42] A. Dabholkar and J.A. Harvey, "Nonrenormalization of the superstring tension", *Phys. Rev. Lett.* **63** (1989) 478.

[43] G.L. Cardoso, B. de Wit, J. Käppeli and T. Mohaupt, "Asymptotic degeneracy of dyonic $N = 4$ string states and black hole entropy", *JHEP* **12** (2004) 075, `hep-th/0412287`.

[44] G.L. Cardoso, B. de Wit and T. Mohaupt, "Macroscopic entropy formulae and non-holomorphic corrections for supersymmetric black holes", *Nucl. Phys.* **B567** (2000) 87–110, `hep-th/9906094`.

[45] E. Witten, "Quantum background independence in string theory", `hep-th/9306122`.

[46] D. S. Freed, "Special Kähler manifolds", *Comm. Math. Phys.* **133** (1999) 31–52, `hep-th/9712042` .

[47] D.V. Alekseevky, V. Cortés and C. Devchand, "Special complex manifolds", *Jour. Geom. Phys.* **42** (2002) 85, `math.DG/9910091`.

[48] N.J. Hitchin, "The moduli space of complex Lagrangian submanifolds", *Asian. J. Math.* **3** (1999) 77, `math.DG/9901069`.

[49] V. Cortés, "A holomorphic representation formula for parabolic hyperspheres", in: B. Opozda, U. Simon and M. Wiehe (eds.), *PDEs, Submanifolds and Affine Differential Geometry*, Banach Center Publications, Warsaw (2000), `math.dg/0107037`.

[50] A. Dabholkar, R. Kallosh and A. Maloney, "A stringy cloak for a classical singularity", *JHEP* **0412** (2004) 059, `hep-th/0410076`.

[51] A. Sen, "How does a fundamental string stretch its horizon?", *JHEP* **0505** (2005) 059, `hep-th/0411255`.

[52] V. Hubeny, A. Maloney and M. Rangamani, "String-corrected black holes", *JHEP* **0505** (2005) 035, `hep-th/0411272`.

[53] S. Chaudhuri, G. Hockney and J.D. Lykken, "Maximally supersymmetric string theories in $D < 10$", *Phys. Rev. Lett.* **75** (1995) 2264, `hep-th/9505054`.

[54] A. Sen, "Black holes, elementary strings and holomorphic anomaly", *JHEP* **0507** (2005) 063, `hep-th/0502126`.

[55] D.P. Jatkar and A. Sen, "Dyon spectrum in CHL models", `hep-th/0510147`.

[56] J.A. Harvey and G.W. Moore, "Fivebrane instantons and R^2 couplings in $N = 4$ string theory", *Phys. Rev.* **D57** (1998) 2323–2328, `hep-th/9610237`.

[57] A. Gregori, E. Kiritsis, C. Kounnas, N.A. Obers, P.M. Petropoulos and B. Pioline, "R^2 Corrections and non-perturbative dualities of $N = 4$ string ground states", *Nucl. Phys.* **B510** (1998) 423, `hep-th/9708062`.

[58] D. Shih and X. Yin, "Exact black hole degeneracies and the topological string", `hep-th/0508174`.

[59] S. Parvizi and A. Tavanfar, "Minimal redefinition of the OSV ensemble", `hep-th/0508231`.

[60] R. Dijkgraaf, E. Verlinde and H. Verlinde, "Counting dyons in $N = 4$ string theory", *Nucl. Phys.* **B484** (1997) 543–561, `hep-th/9607026`.

[61] S. Shih, A. Strominger and X. Yin, "Recounting dyons in $N = 4$ string theory", `hep-th/0505094`.

[62] A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, "Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity", *Nucl. Phys.* **B444** (1995) 92, `hep-th/9502072`.

[63] B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, "Perturbative couplings of vector multiplets in $N = 2$ heterotic string vacua", *Nucl. Phys.* **B451** (1995) 53, `hep-th/9504006`.